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# Double scaling limits in gauge theories and matrix models

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Abstract: We show that $\mathcal{N}=1$ gauge theories with an adjoint chiral multiplet admit a wide class of large-$N$ double-scaling limits where $N$ is taken to infinity in a way coordinated with a tuning of the bare superpotential. The tuning is such that the theory is near an Argyres-Douglas-type singularity where a set of non-local dibaryons becomes massless in conjunction with a set of confining strings becoming tensionless. The doubly-scaled theory consists of two decoupled sectors, one whose spectrum and interactions follow the usual large-$N$ scaling whilst the other has light states of fixed mass in the large-$N$ limit which subvert the usual large-$N$ scaling and lead to an interacting theory in the limit. $F$-term properties of this interacting sector can be calculated using a Dijkgraaf-Vafa matrix model and in this context the double-scaling limit is precisely the kind investigated in the "old matrix model" to describe two-dimensional gravity coupled to $c<1$ conformal field theories. In particular, the old matrix model double-scaling limit describes a sector of a gauge theory with a mass gap and light meson-like composite states, the approximate Goldstone boson of superconformal invariance, with a mass which is fixed in the double-scaling limit. Consequently, the gravitational $F$-terms in these cases satisfy the string equation of the KdV hierarchy.

Keywords: Supersymmetry and Duality, Topological Strings, 1/N Expansion.

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## 1. Introduction

't Hooft argued many years ago [1] that the perturbative expansion of theories involving matrices has an interesting interpretation when $N$, the size of the matrices, becomes large. The idea is to first re-write the perturbative expansion in $g_s$, the coupling, and $N$, in terms of $S = g_s N$, the "'t Hooft Coupling", and $g_s$. The Feynman graphs of the theory can be sorted on the basis of the topology of an imaginary two-dimensional surface on which the diagrams can be drawn. In terms of $(S, g_s)$, the explicit $g_s$ dependence of a graph is determined solely by the topology of the imaginary surface: if $g$ is the genus then the dependence is $g_s^{2g-2}$. At a given genus, there is then a perturbative series in the 't Hooft coupling $S$.

An example of such matrix theories are Yang-Mills theories. These ideas led 't Hooft to suggest that Yang-Mills theories may have an alternative description in which the imaginary two-dimensional surfaces become physically realized as the world-sheets of a string theory. In this picture, $g_s = g_{\rm YM}^2$ becomes the *string coupling* weighting each handle in string world-sheet perturbation theory, so that the string theory is weakly coupled when $N$ is large. Quite how gauge theories are equivalent to a theory of strings has taken many years to understand in a concrete way.



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The topological expansion of matrix field theories is quite general and applies, for example, to simple matrix theories (zero-dimensional field theories), where there is no spacetime. The fact that such simple theories can give rise to two-dimensional surfaces has been an idea of great import. In the "old matrix model" epoch the idea was to use matrix models to describe models of two-dimensional gravity coupled to various matter systems. Here, once again the surfaces arise in an auxiliary way from the expansion of the matrix model in the string coupling. In these theories the way that the Feynman graphs give rise to a two-dimensional surface is completely transparent. The 't Hooft expansion of, say, the free energy has the form

$$F = \sum_{g=0}^{\infty} F_g(\xi) g_s^{2g-2} \ . \tag{1.1}$$

Here, $\xi$ label a general set of parameters in the theory (suitably scaled in $N$), including the 't Hooft coupling. The surfaces arise when one takes the large-$N$ limit with fixed 't Hooft coupling, as before, but, in addition, one takes a parameter $\xi$ to a critical value. At the critical point $\xi = \xi_0$, the contributions from a given genus $F_g(\xi)$ diverge in a particular way:

$$F_g(\xi) \sim |\xi - \xi_0|^{\eta(2-2g)} \ , \tag{1.2}$$

where $\eta$ is a critical exponent which characterizes the universality of the critical point. Intuitively, near the critical point the Feynman graphs of a given genus become very dense and describe a surface. The coordinated, or double-scaling limit, involves taking [2]

$$N \to \infty \ , \qquad \xi \to \xi_0 \ , \qquad \Delta = N|\xi - \xi_0|^{\eta} = \text{fixed} \ . \tag{1.3}$$

In this limit there is an effective string coupling

$$(g_s)_{\text{eff}} \sim \Delta^{-1} \tag{1.4}$$

which weights the contributions from different genera.

What makes the old matrix model story so compelling is the ease at which the two-dimensional surfaces arise. This is not so clear in Yang-Mills theories where the surfaces are thought to be associated to a dual string description. Realizing this picture explicitly was brilliantly achieved by Maldacena in the case of maximally supersymmetric $\mathcal{N} = 4$ gauge theories [3]. In this case the dual string theory is a ten-dimensional Type IIB superstring on the space $AdS_5 \times S^5$ with background Ramond-Ramond (RR) flux. Unfortunately such string theories have proved very difficult to solve because of the presence of the RR flux. It would be very useful to have a situation where the string theory appeared in a very simple way as in the old matrix model. What we have in mind is a realization of a double-scaling limit in a Yang-Mills theory where $N$ is taken to infinity as certain couplings in a bare potential are taken to critical values in such a way that the string dual naturally arises. Double-scaling limits of this kind have been recently studied in [4] for a class of $\mathcal{N} = 1$ gauge theories in a *partially confining* phase. There it was proposed that they admit a dual description in terms of a non-critical string theory of the type first introduced in [5]. The worldsheet theory corresponding to this string background is exactly solvable and there are



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no RR fluxes. The gauge theories involve a gauge group $G = \mathrm{U}(N)$ which is confined down to an abelian group. The theories have an $\mathcal{N} = 1$ vector multiplet and an adjoint-valued chiral multiplet $\Phi$ (precisely the fields of an $\mathcal{N} = 2$ theory).[1] There is an explicit bare superpotential

$$W_{\text{bare}}(\Phi) = \operatorname{Tr} W(\Phi)\,, \qquad W(x) = N \sum_{i=1}^{\ell+1} \frac{g_i}{i} x^i \ . \tag{1.5}$$

The prefactor ensures conventional large-$N$ scaling. Classically, each of the eigenvalues of the scalar component of $\Phi$ can be one of the $\ell$ critical points of $W(x)$,

$$W'(x) = N\varepsilon \prod_{i=1}^{\ell} (x - a_i)\,, \qquad \varepsilon \equiv g_{\ell+1} \ . \tag{1.6}$$

We define the multiplicity of eigenvalues at the critical point $a_i$ as $N_i \geq 0$. At least in an appropriate limit, this leads to a classical Higgs effect $\mathrm{U}(N) \to \prod_{i=1}^{\ell} \mathrm{U}(N_i)$, where $\sum_{i=1}^{\ell} N_i = N$. Each $\mathrm{U}(N_i)$ will then subsequently confine to leave a $\mathrm{U}(1)$ factor in the IR. We will denote by $s$ the number of $N_i > 0$, so that the IR gauge group is $\mathrm{U}(1)^s$. Of course, such a picture of a classical Higgs effect plus confinement is not expected to be valid for all the values of the parameters but is a useful limit to have in mind. In particular, vacua with the same rank of IR group, but with different "filling fractions" $N_i/N$, are known, under certain circumstances, to be continuously connected in the space of superpotentials [9, 10].

The spectrum of these partially confining theories is expected to be rather complicated and since there is only $\mathcal{N} = 1$ supersymmetry and no notion of a BPS state, we have little analytic control over particle masses. However, we expect that for a generic vacuum the conventional large-$N$ picture applies (see for example [11]): as well as the massless abelian sector, there will be a tower of glueballs and "dimesons", formed by one "diquark" $Q_{rs}$ transforming in the $(N_r, N_s)$ representation of $\mathrm{U}(N_r) \times \mathrm{U}(N_s)$ and the corresponding anti-diquark $\bar{Q}_{rs}$, whose masses are $N$ independent. There will also be dibaryons, whose masses scale like $N$. The interactions of the glueballs and mesons are suppressed by $1/N$,[2] whereas the interactions between the baryons and mesons are not suppressed. However, if there is a point $\xi_0$ in the space of parameters $\{g_i\}$ where some state becomes massless then the usual large-$N$ picture can break down. These states are baryonic (or composites of these) with a mass that generically goes like $N$, but near the critical point $\xi_0$ one finds that $M \sim N|\xi - \xi_0|^\eta$ for some exponent $\eta$. In the double-scaling limit in which $N \to \infty$ and $\xi \to \xi_0$ such that $M$ is fixed, the theory decomposes into two sectors $\mathcal{H} \to \mathcal{H}_- \times \mathcal{H}_+$. For the sector $\mathcal{H}_-$, the usual rules of the large-$N$ expansion are modified by diagrams involving the state of mass $M$ running around in loops. As described in [4] there is now an effective string coupling as in (1.4). The sector $\mathcal{H}_+$, on the other hand, decouples and is bound by

[1] The $\beta$-deformation of $\mathcal{N} = 4$ Super Yang-Mills also has a partially confining phase that has been studied in [6 – 8]. A double-scaling limit and the relative non-critical string dual have also been proposed in [4].

[2] The interactions between mesons are suppressed by integer powers of $1/N$ instead of the usual $1/\sqrt{N}$ because these states are formed by one "diquark" $Q_{rs}$ transforming in the $(N_r, N_s)$ representation of $\mathrm{U}(N_r) \times \mathrm{U}(N_s)$ and the corresponding anti-diquark $\bar{Q}_{rs}$. Therefore the three-point coupling of these "dimesons" gains a suppression factor $\sim 1/\sqrt{N}$ from each of the two groups, $N_r \sim N_s \sim N$ [4].



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conventional large-$N$ reasoning (and hence is not interesting). What is remarkable is that there is a natural candidate for the string dual to the sector $\mathcal{H}_-$, namely a non-critical string background of the type introduced in [12]. These non-critical backgrounds were originally studied as holographic duals to certain Little String Theories (LST) and double-scaled LST [13 − 16]. In the low-energy limit these LST reduce to 4d $\mathcal{N} = 2$ theories in the proximity of Argyres-Douglas singularities which are non-trivial superconformal field theories [17 − 19]. The fact that these backgrounds preserve $\mathcal{N} = 2$ supersymmetry seems to contradict the proposed duality. Remarkably, the $F$-terms of the theory are consistent with this supersymmetry enhancement in the $\mathcal{H}_-$ sector [4]. It is interesting to try to extend this kind of duality and search for more general classes of double-scaling limits in supersymmetric gauge theories. In this paper, we will achieve this aim and we will show in many cases that these double-scaling limits exist and that the Hilbert space of the theory still exhibits the above splitting into two decoupled sectors. As before, the sector $\mathcal{H}_-$ has non-trivial dynamics in the double-scaling limit weighted by the effective string coupling (1.4). However, in the generic situation we consider, this sector only has $\mathcal{N} = 1$ supersymmetry since the effects of the superpotential do not vanish in the limit. This is different from the case studied in [4], where the superpotential vanished in the $\mathcal{H}_-$ sector, which is consistent with the enhancement to $\mathcal{N} = 2$ supersymmetry. Consequently, what is missing from our analysis is the precise identification of the dual string theory itself.[3] We will also find examples where the sector $\mathcal{H}_-$ does not contain any U(1) group but has a mass gap and light meson-like composite states.

So our focus is on large-$N$ double-scaling limits in supersymmetric gauge theories. As we have argued, in order to have such a limit we need a scenario in which there are states which we can tune to have a finite mass as $N \to \infty$. The question is how can we verify that such double-scaling limits actually exist because we do not generally have analytic control over the masses of particles in an $\mathcal{N} = 1$ theory? The answer is that, under certain circumstances, we can see the effects of a nearly massless particle in various $F$-terms and these can often be calculated exactly. For instance, if the particle is charged under a U(1) gauge symmetry that survives in the IR, then we will see the effects of the light particle in the associated renormalization of the U(1) coupling. This was one of the strategies employed in [4] to explore these double-scaling limits. In the present work, we will find situations where the light states are not charged under any U(1) in the IR. So in order to "see" these kinds of states we will have to investigate other $F$-term couplings of the gauge theory. The idea is to consider such couplings that arise when the gauge theory is coupled to supergravity. In that case there is a whole set of gravitational $F$-terms [21, 22]. There are two series of terms that interest us

$$\Gamma_1 = \sum_{g=0}^{\infty} \int d^4x\, d^2\theta\, (F_{\alpha\beta}F^{\alpha\beta})^g \sum_{i=1}^{s} N_i \frac{\partial F_g(S)}{\partial S_i} \tag{1.7}$$

[3] A construction of four-dimensional $\mathcal{N} = 1$ non-critical strings dual to $\mathcal{N} = 1$ quiver gauge theories has recently appeared in [20].



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and

$$\Gamma_2 = \sum_{g=1}^{\infty} \int d^4x\, d^2\theta\, \mathcal{W}_{\alpha\beta\gamma}\mathcal{W}^{\alpha\beta\gamma}(F_{\delta\epsilon}F^{\delta\epsilon})^{g-1}F_g(S) \ . \tag{1.8}$$

Here, $\mathcal{W}_{\alpha\beta\gamma}$ and $F_{\alpha\beta}$ are the $\mathcal{N}=2$ gravitino and gravi-photon superfields of the gravity sector, respectively. The $F_g(S)$ are the gauge theory objects: functions of the $s$ glueball superfields $\{S_i\}$. Note that the $g=0$ term of $\Gamma_1$ is the ordinary glueball superpotential:

$$W_{\text{gb}} = \sum_{i=1}^{s} N_i \frac{\partial F_0(S)}{\partial S_i} \,, \tag{1.9}$$

which determines the vacua of the theory.[4]

In the usual large-$N$ limit, the contribution from a genus $g$ graph is suppressed by $N^{2-2g}$. However, if there is a scenario as described above where a state is becoming massless then this suppression can be subverted. The effect of such a state on $F_g$ can be crudely estimated by considering the light state propagating around a 1-loop graph with $g$ insertions [4, 24]. This suggests $F_g \sim M^{2-2g}$, which together with the identification $M \sim \Delta$ means that the gravitational $F$-term $F_g$ behaves just like the genus $g$ free energy of a matrix model as in (1.2). The conclusion is that the quantities $F_g$ appearing in the gravitational $F$-terms provide another signal of a double-scaling limit. Of course it is no accident that these couplings behave like the free energy of a matrix model since they can be calculated by just such a matrix model as described by Dijkgraaf and Vafa [25]. This matrix model is defined in terms of an $\hat{N}\times\hat{N}$ matrix $\hat{\Phi}$ ($\hat{N}$ here is not to be confused with $N$ of the field theory) with the bare superpotential in (1.5):

$$\exp \sum_{g=0}^{\infty} F_g g_s^{2g-2} = \int d\hat{\Phi}\, \exp\Big( - g_s^{-1} \text{Tr}\, W(\hat{\Phi})\Big) \ . \tag{1.10}$$

The integral is to be understood as a saddle-point expansion around a critical point where $\hat{N}_i$ of the eigenvalues sit in the critical point $a_i$ (but only if the associated $N_i > 0$) and by definition the glueball superfields are identified with the quantities

$$S_i = g_s \hat{N}_i \,, \qquad S = \sum_{i=1}^{s} S_i = g_s \hat{N} \tag{1.11}$$

in the matrix model. It is worth remarking that in the matrix model we work at large $\hat{N}$ and that this procedure would yield exact results for any value of $N$. But, of course, we also want to take $N$ to be large in the double-scaling limit.

It should now be apparent that the double-scaling limit of the gravitational $F$-terms (1.7) (1.8) in the gauge theory corresponds precisely to the notion of a double-scaling limit in the "old" matrix model and in certain cases we can use standard matrix model technology to investigate the phenomenon.

[4]We are assuming that the VEVs of the supergravity fields vanish otherwise we have to keep all the terms in $\Gamma_1$ in the superpotential as in [23].

The results presented in this paper have some overlap with earlier work. The idea that the standard large-$N$ expansion breaks down near the critical points in the moduli/parameter space of four-dimensional gauge theories has been emphasized in a series of papers by Ferrari [26 – 28], which also contain proposals for double-scaling limits. The four-dimensional interpretation of double-scaling in the Dijkgraaf-Vafa matrix model has also been discussed before in [25, 26].

The plan of the paper is as follows. In section 2, we will give a general description of double-scaling limits and explain how they can be realized on-shell. In section 3, we will study the exact $F$-term couplings of the various models and deduce that in the double-scaling limit the Hilbert space of theory decomposes into two decoupled sectors $\mathcal{H}_+$ and $\mathcal{H}_-$, where $\mathcal{H}_-$ keeps finite interactions in the limit. In section 4, we will analyze the behaviour of the higher genus terms of the matrix model free energy $F_g$ at the critical point and show that $F_g \sim \Delta^{2-2g}$. In section 5, we will discuss in detail the case where the near-critical curve has double points. We will argue that in the critical limit where a double point collides with a branch point a meson-like state, which can be thought of as the bound state of two dibaryons, becomes massless.

## 2. Engineering double-scaling limits in the matrix model

The central object in matrix model theory is the resolvent

$$\omega(x) = \frac{1}{\hat{N}}\mathrm{Tr}\,\frac{1}{x-\hat{\Phi}} \ . \tag{2.1}$$

At leading order in the $1/\hat{N}$ expansion, $\omega(x)$ is valued on the spectral curve $\Sigma$, a hyperelliptic Riemann surface

$$y^2 = \frac{1}{(N\varepsilon)^2}\big(W'(x)^2 + f_{\ell-1}(x)\big) \ . \tag{2.2}$$

The numerical prefactor is chosen for convenience. In terms of this curve

$$\omega(x) = \frac{1}{2S}\big(W'(x) - N\varepsilon y(x)\big) \ . \tag{2.3}$$

In (2.2), $f_{\ell-1}(x)$ is a polynomial of order $\ell-1$ whose $\ell$ coefficients are moduli that are determined by the $S_i$. In general, the spectral curve can be viewed as a double-cover of the complex plane connected by $\ell$ cuts. For the saddle-point of interest only $s$ of the cuts are opened and so only $s$ of the moduli $f_{\ell-1}(x)$ can vary. Consequently $y(x)$ has $2s$ branch points and $\ell - s$ zeros:[5]

$$\Sigma: \qquad y^2 = Z_m(x)^2\sigma_{2s}(x) \tag{2.4}$$

where $\ell = m + s$ and

$$Z_m(x) = \prod_{j=1}^{m}(x-z_j)\,, \qquad \sigma_{2s}(x) = \prod_{j=1}^{2s}(x-\sigma_j) \ . \tag{2.5}$$

[5] Occasionally, for clarity, we indicate the order of a polynomial by a subscript.

The remaining moduli are related to the $s$ parameters $\{S_i\}$ by (1.11)

$$S_i = g_s \hat{N}_i = N\varepsilon \oint_{A_i} y\, dx\,, \tag{2.6}$$

where the cycle $A_i$ encircles the cut which opens out around the critical point $a_i$ of $W(x)$.

Experience with the old matrix model teaches us that double-scaling limits can exist when the parameters in the potential are varied in such a way that combinations of branch and double points come together. In the neighbourhood of such a critical point,[6]

$$y^2 \longrightarrow C Z_m(x)^2 B_n(x)\,, \qquad B_n(x) = \prod_{i=1}^{n}(x - b_i)\,, \tag{2.7}$$

where $z_j, b_i \to x_0$, which we can take, without loss of generality, to be $x_0 = 0$. The double-scaling limit involves first taking $a \to 0$

$$x = a\tilde{x}\,, \qquad z_i = a\tilde{z}_i\,, \qquad b_j = a\tilde{b}_j \tag{2.8}$$

while keeping tilded quantities fixed. In the limit, we can define the near-critical curve $\Sigma_-$:[7]

$$\Sigma_- : \qquad y_-^2 = \tilde{Z}_m(\tilde{x})^2 \tilde{B}_n(\tilde{x})\ . \tag{2.9}$$

We will argue, generalizing [4], that in the limit $a \to 0$, in its sense as a complex manifold, the curve $\Sigma$ factorizes as $\Sigma_- \cup \Sigma_+$. The complement to the near-critical curve is of the form

$$\Sigma_+ : \qquad y_+^2 = x^{2m+n} C_{2s-n}(x)\ . \tag{2.10}$$

where $C_{2s-n}(x)$ is regular at $a = 0$.

In the $a \to 0$ limit, we will show in section 4 that the genus $g$ free energy gets a dominant contribution from $\Sigma_-$ of the form

$$F_g \sim \left(N a^{(m+n/2+1)}\right)^{2-2g}\ . \tag{2.11}$$

Note that in this equation $N$ is the one from the field theory and not the matrix model $\hat{N}$. This motivates us to define the double-scaling limit

$$a \to 0\,, \qquad N \to \infty\,, \qquad \Delta \equiv N a^{m+n/2+1} = \text{const}\ . \tag{2.12}$$

Moreover, the most singular terms in $a$ in (2.11) depend only on the near-critical curve (2.9) in a universal way.

### 2.1 Engineering the double-scaling limit on-shell

However, there is still an important issue to address. In the context of supersymmetric gauge theories, the moduli $\{S_i\}$ are fixed by extremizing the glueball superpotential (1.9). It is not, *a priori*, clear whether a double-scaling limit can be reached whilst simultaneously

[6] We have chosen for convenience to take all the double zeros $\{z_j\}$ into the critical region.

[7] For polynomials, we use the notation $\tilde{f}(\tilde{x}) = \prod_i(\tilde{x} - \tilde{f}_i)$, where $f(x) = \prod_i(x - f_i)$, $x = a\tilde{x}$ and $f_i = a\tilde{f}_i$.



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being on-shell with respect to the glueball superpotential. We now address this issue and show that suitable choices of the coupling constants $\{g_i\}$ do indeed allow for a double-scaling limit on-shell with respect to the glueball superpotential. In general, the potentials required are non-minimal. However, this is irrelevant for extracting the universal behaviour that only depends on the near-critical curve (2.7).

So the problem before us is to show that the critical point can be reached simultaneously with being at a critical point of the glueball superpotential. It is rather difficult to find the critical points of the latter directly. Fortunately another more tractable method consists of comparing the matrix model spectral curve (2.2), the "$\mathcal{N} = 1$ curve", with the Seiberg-Witten curve of the underlying $\mathcal{N} = 2$ theory that results when the potential vanishes. The latter has the form

$$y_{\rm SW}^2 = P_N(x)^2 - 4\Lambda^{2N}\,, \tag{2.13}$$

where $P_N(x) = \prod_{i=1}^N (x - \phi_i)$. Here, $\{\phi_i\}$ are a set of coordinates on the Coulomb branch of the $\mathcal{N} = 2$ theory and $\Lambda$ is the usual scale of strong-coupling effects in the $\mathcal{N} = 2$ theory.

When the $\mathcal{N} = 2$ theory is deformed by addition of the superpotential (1.5), it can be shown that a vacuum exists when the Seiberg-Witten curve and the $\mathcal{N} = 1$ curve represent the same underlying Riemann surface [10, 29, 30]. In concrete terms this means that, on-shell,

$$\begin{aligned} y_{\rm SW}^2 &= P_N(x)^2 - 4\Lambda^{2N} = H_{N-s}(x)^2\sigma_{2s}(x) \\ y^2 &= \frac{1}{(N\varepsilon)^2}\big(W'(x)^2 + f_{s+m-1}(x)\big) = Z_m(x)^2\sigma_{2s}(x)\,, \end{aligned} \tag{2.14}$$

In these equations, $H_{N-s}(x)$, $\sigma_{2s}(x)$, $Z_m(x)$ are polynomials of the indicated order, and we choose (in order to remove some redundancies)

$$H_{N-s}(x) = x^{N-s} + \cdots\,, \qquad \sigma_{2s}(x) = x^{2s} + \cdots\,, \qquad Z_m(x) = x^m + \cdots\,. \tag{2.15}$$

Both curves describe the same underlying Riemann surface, namely the reduced curve of genus $s-1$ which is a hyper-elliptic double-cover of the complex plane with $s$ cuts. All-in-all there are $2(N+l)$ equations for the same number of unknowns in $\{P, H, \sigma, Z, f\}$. There are many solutions to these equations and we can make contact with the description of the vacua in section 1 by taking the classical limit $\Lambda \to 0$; whence

$$P_N(x) \to \prod_{i=1}^{\ell} (x - a_i)^{N_i}\,, \qquad \sum_{i=1}^{\ell} N_i = N\,, \tag{2.16}$$

so $N_i$ of the eigenvalues of the Higgs field classically lie at the critical point $a_i$ of $W(x)$. Quantum effects then have the effect of opening the points $a_i$ into cuts (if $N_i > 0$). The number of $N_i > 0$, *i.e* the number of cuts, is equal to $s = \ell - m$.

We now to turn to explicit solutions of (2.14). The method we shall adopt is to first find solutions for a $\mathrm{U}(p)$ gauge theory and then apply the "multiplication by $N/p$ map" [30], with $N/p$ integer. This will yield a solution for a $\mathrm{U}(N)$ gauge group and will allow to take a large $N$ limit with $p$ fixed.

### 2.2 No double points

We now describe how to engineer the case where the near critical curve (2.7) has no double points, so $m = 0$. This is the situation considered in [4, 31, 32]. In this case, we first consider the consistency conditions (2.14) for a $\mathrm{U}(p = n)$ gauge theory with $W(x)$ of order $\ell = n + 1$. In this case, (2.14) are trivially satisfied with

$$W'(x) = N\varepsilon P_n(x)\,, \qquad f_{n-1}(x) = -4N^2\varepsilon^2\Lambda^{2n}\ . \tag{2.17}$$

Notice that with our minimal choice of potential, the on-shell curve actually implies that $S = 0$ since the coefficient of $x^{n-1}$ in $f_{n-1}(x)$ vanishes and so the resolvent falls faster than $1/x$ at infinity. This, of course, is pathological from the point-of-view of the old matrix model and may be remedied by using a non-minimal potential with extra branch points or double points outside the critical region. However, in the holomorphic context in which we are working, having $S = 0$ is perfectly acceptable and we stick with it. The on-shell curve consists of an $n$-cut hyperelliptic curve and one can verify, by taking the classical limit, that $N_i = 1$, $i = 1, \ldots, n$. The double-scaling limit involves a situation where $n$ branch points, one from each of the cuts, come together. This can be arranged by having

$$W'(x) = N\varepsilon\big(B_n(x) + 2\Lambda^n\big)\,, \qquad B_n(x) = \prod_{j=1}^{n}(x - b_j) \tag{2.18}$$

and then taking the limit (2.8). In this case, the near-critical curve $\Sigma_-$ (2.9) is of the form

$$y_-^2 = \tilde{B}_n(\tilde{x})\ . \tag{2.19}$$

The important point is that we can tune to the critical region whilst keeping the theory on-shell with respect to the glueball superpotential by simply changing the parameters $\{b_j\}$ which appear in the potential.

Now that we have found a suitable vacuum of a $\mathrm{U}(n)$ theory, we now lift this to a $\mathrm{U}(N)$ theory with the multiplication by $N/n$ map [30]. Under this map, the $\mathcal{N} = 1$ curve remains intact, including the potential $W(x)$ whilst the Seiberg-Witten curve of the $\mathrm{U}(N)$ theory is

$$y_{\rm SW}^2 = P_N(x)^2 - 4\Lambda^{2N} = \Lambda^{2(N-n)}\mathcal{U}_{\frac{N}{n}-1}\Big(\frac{P_n(x)}{2\Lambda^n}\Big)^2\big(P_n(x)^2 - 4\Lambda^{2n}\big)\,, \tag{2.20}$$

where $\mathcal{U}_{\frac{N}{n}-1}(x)$ is a Chebishev polynomial of the second kind. The vacuum of the $\mathrm{U}(N)$ theory has $N_i = N/n$, $i = 1, \ldots, n$.

Notice that in the near critical region the Seiberg-Witten curve is identical to $\Sigma_-$, up to a rescaling:

$$y_{\rm SW}^2 \longrightarrow \Big(\frac{2N}{n}\Big)^2\Lambda^{2N-n}B_n(x)\ . \tag{2.21}$$

This is simply a reflection of the observation of [4] that the decoupled sector has enhanced $\mathcal{N} = 2$ supersymmetry. Moreover, if $C$ is a cycle which is vanishing as $a \to 0$ then the integral of the Seiberg-Witten differential around $C$, which gives the mass of a BPS state carrying electric and magnetic charges in the theory, becomes

$$\oint_C \frac{xP'_N(x)\,dx}{y_{\rm SW}} \longrightarrow -\Lambda^{-n/2} N a^{n/2+1} \oint_C y_-\,d\tilde{x} \ . \tag{2.22}$$

Notice that in the double-scaling limit (2.12) (with $m = 0$) the mass of the state is fixed. This state is a dibaryon that carries electric and magnetic charges of the IR gauge group. In the double-scaling limit, therefore, a set of mutually non-local dibaryons become very light.[8] In fact, the Seiberg-Witten curve at the critical point, $a = 0$, has the form

$$y_{\rm SW}^2 = 4\Big(\frac{N}{n}\Big)^2 \Lambda^{2N-n} x^n \ , \tag{2.23}$$

which describes a $\mathbf{Z}_n$ or $A_{n-1}$ Argyres-Douglas singularity [17–19].

### 2.3 With double points

For the case with double points, we cannot simply take two of the branch points $\{b_j\}$ in (2.18) above to be the same. If we simply did that then the zero of the Seiberg-Witten curve, by which we mean a zero of the polynomial $H_{N-s}$ in (2.14), would also be a zero of the $\mathcal{N} = 1$ curve as well. By the analysis of [29], this would imply that the condensate of the associated massless dibaryon would be vanishing. On the contrary, if the zero of the Seiberg-Witten curve were not a zero of the $\mathcal{N} = 1$ curve, the putative massless dibaryon would be condensed and the dual U(1) would be confined. We need to arrange the situation so that any zero of the Seiberg-Witten curve is not simultaneously a zero of the $\mathcal{N} = 1$ curve.

A suitable $\mathcal{N} = 1$ curve which reduces to (2.7) in the near-critical region is

$$y^2 = Z_m(x)^2 B_n(x)\big(B_n(x)H_r(x)^2 + 4\Lambda^{2r+n}\big) \ . \tag{2.24}$$

In this case, we have $\ell = m + n + r$, $s = n + r$ and

$$W'(x) = N\varepsilon Z_m(x)B_n(x)H_r(x) \,, \qquad f_{\ell-1}(x) = 4N^2\varepsilon^2\Lambda^{2r+n} Z_m(x)^2 B_n(x) \ . \tag{2.25}$$

Notice that in order that $f_{\ell-1}(x)$ has order less than $\ell$ we require $r > m$. The curve (2.24) is actually on-shell with respect to the Seiberg-Witten curve of a $\mathrm{U}(2r + n)$ theory with

$$P_{2r+n}(x) = H_r(x)^2 B_n(x) + 2\Lambda^{2r+n} \ . \tag{2.26}$$

In the classical limit, we have two eigenvalues at each of the zeros of $H_r(x)$ and one in each of the zeros of $B_n(x)$. Once again we can employ the multiplication map (2.20) (with $n$ replaced by $2r + n$) to find the vacuum of the $\mathrm{U}(N)$ theory we are after.

Notice that the double points of the Seiberg-Witten curve $\{h_i\}$ are not generally zeros of the curve (2.24), which means that the associated dyons are condensed. The near-critical curve $\Sigma_-$ in this case is

$$y_-^2 = \tilde{Z}_m(\tilde{x})^2 \tilde{B}_n(\tilde{x}) \,, \tag{2.27}$$

[8] For $n = 2$ there is only a single light dibaryon.

while in the near-critical region the Seiberg-Witten curve becomes

$$y_{\rm SW}^2 \longrightarrow 4\Big(\frac{N}{n+2r}\Big)^2 \Lambda^{2N-2r-n} H_r(0)^2 B_n(x) \ . \tag{2.28}$$

where we assumed that the zeros of $H_r(x)$ lie outside the critical region. In this case, the integral of the Seiberg-Witten differential around a vanishing cycle diverges in the double-scaling limit:

$$\oint_C \frac{xP_N'(x)\,dx}{y_{\rm SW}} \sim Na^{n/2+1} = \Delta a^{-m} \to \infty \ . \tag{2.29}$$

So in contrast to the case with no double points, the dibaryon states are very heavy. In addition, the dyon condensate associated to the zero $h_i$ of $H_r(x)$ is given by an exact formula [29]

$$\langle m_i\tilde m_i\rangle = N\varepsilon y(h_i) \sim N \to \infty\,, \tag{2.30}$$

where we have assumed that $h_i$ stays fixed as $a \to 0$. So in the double-scaling limit the value of the condensate and hence the confinement scale in the dual U(1), or string tension, occurs at a very high mass scale.

We now have a puzzle. How can there be an interesting double-scaling limit in the gauge theory in this case if there are no light dibaryon as in the previous example? The answer is that there are other light mesonic states in the theory with a mass $\sim \Delta$ that we will identify in section 5.

Notice that contrary to our choice above, if we scale $h_i \to 0$ as $a \to 0$ then the tensions of the confining strings vanish and the theory is at an $\mathcal{N}=1$ superconformal fixed point in the infra-red corresponding to one of the $\mathcal{N}=1$ Argyres-Douglas-type singularities described in [31]. As the double points of the Seiberg-Witten curve $h_i$ move away from 0 the associated dyons condense and the superconformal invariance is spontaneously broken. The resulting Goldstone modes will play an important rôle in the ensuing story and resolve the puzzle alluded to above.

## 3. The double-scaling limit of $F$-terms

Before we consider the $a \to 0$ limit of the free energy, it is useful to the consider this limit for other $F$-terms in the low-energy effective action. The effective action is written in terms of chiral superfields $S_l$ and $w_{\alpha l}$ which are defined as gauge-invariant single-trace operators [33]

$$\begin{aligned} S_l &= -\frac{1}{2\pi i}\oint_{A_l} dx\,\frac{1}{32\pi^2}{\rm Tr}_N\left[\frac{W_\alpha W^\alpha}{x-\Phi}\right]\,, \\ w_{\alpha l} &= \frac{1}{2\pi i}\oint_{A_l} dx\,\frac{1}{4\pi}{\rm Tr}_N\left[\frac{W_\alpha}{x-\Phi}\right]\ . \end{aligned} \tag{3.1}$$

It will also be convenient to define component fields for each of these superfields,

$$S_l = s_l + \theta_\alpha \chi_l^\alpha + \cdots\,, \qquad w_{\alpha l} = \lambda_{\alpha l} + \theta_\beta f_{\alpha l}^\beta + \cdots \ . \tag{3.2}$$

The component fields, $s_l$ and $f_l$ are bosonic single trace operators whilst $\chi_l$ and $\lambda_l$ are fermionic single trace operators. In the large-$N$ limit, these operators should create bosonic and fermionic colour-singlet single particle states respectively. It is instructive to consider the interaction vertices for these fields contained in the $F$-term effective action whose general form is given by [25, 34, 35]

$$\mathcal{L}_F = \mathrm{Im}\left[\int d^2\theta \left(W_{\mathrm{gb}} + W_{\mathrm{eff}}^{(2)}\right)\right] \tag{3.3}$$

where

$$W_{\mathrm{eff}}^{(2)} = \frac{1}{2}\sum_{k,l} \frac{\partial^2 F_0}{\partial S_k \partial S_l}\, w_{\alpha k} w_l^{\alpha} \ . \tag{3.4}$$

Expanding (3.3) in components on-shell, we find terms like

$$\int d^2\theta\, W_{\mathrm{eff}}^{(2)} \supset V_{ij}^{(2)} f_{\alpha\beta}^i f^{\alpha\beta j} + V_{ijk}^{(3)} \chi_\alpha^i f^{\alpha\beta j} \lambda_\beta^k + V_{ijkl}^{(4)} \chi_\alpha^i \chi^{\alpha j} \lambda_\beta^k \lambda^{\beta l}\,, \tag{3.5}$$

where

$$V_{i_1 i_2 \ldots i_L}^{(L)} = \frac{\partial^L F_0}{\partial S_{i_1} \partial S_{i_2} \ldots \partial S_{i_L}} \tag{3.6}$$

for $L = 2, 3, 4$. In the large-$N$ limit, $V^{(L)}$ scales like $N^{2-L}$. We will also consider the 2-point vertex coming from the glueball superpotential

$$\int d^2\theta\, W_{\mathrm{gb}} \supset H_{ij}^{(2)} \chi_\alpha^i \chi^{\alpha j}\,, \tag{3.7}$$

where

$$H_{ij}^{(2)} = \frac{\partial^2 W_{\mathrm{gb}}}{\partial S_i \partial S_j} \ . \tag{3.8}$$

The matrix $H_{ij}^{(2)}$ therefore effectively determines the masses of the chiral multiplets $S_l$. Note that, in the large-$N$ limit, $H^{(2)}$ scales like $N^0$.

We begin by considering the couplings $V_{ij}^{(2)}$ of the low-energy $\mathrm{U}(1)^s$ gauge group. Each of the U(1)'s is associated to one of the glueball fields $S_i$, or equivalently the set of 1-cycles $\{A_i\}$ on $\Sigma$. If we ignore the U(1) associated to the overall 't Hooft coupling $S$, or the cycle $A_\infty = \sum_{i=1}^s A_i$ which can be pulled off to infinity, the couplings of the remaining ones are simply the elements of the period matrix of $\Sigma$. In order to take the $a \to 0$ limit, it is useful to choose a new basis of 1-cycles $\{\tilde{A}_i, \tilde{B}_i\}$, $i = 1, \ldots, s-1$, which is specifically adapted to the factorization $\Sigma \to \Sigma_- \cup \Sigma_+$. The subset of cycles with $i = 1, \ldots, [n/2]$ vanish at the critical point while the cycles $i = [n/2]+1, \ldots, s-1$ are the remaining cycles which have zero intersection with all the vanishing cycles.

If we define the periods on $\Sigma$

$$M_{ij} = \oint_{\tilde{B}_j} \frac{x^{i-1}}{\sqrt{\sigma(x)}}\, dx\,, \qquad N_{ij} = \oint_{\tilde{A}_j} \frac{x^{i-1}}{\sqrt{\sigma(x)}}\, dx\,, \tag{3.9}$$

then the period matrix, in this basis, is simply

$$\Pi = N^{-1} M \ . \tag{3.10}$$

In the appendix, we calculate the $a \to 0$ limit of these matrices. The results are summarized in (A.4) and (A.7). Using these results, we have

$$\Pi \longrightarrow \begin{pmatrix} N_{--}^{-1} M_{--} & N_{--}^{-1} M_{-+}^{(0)} + \mathcal{N} M_{++}^{(0)} \\ 0 & \left(N_{++}^{(0)}\right)^{-1} M_{++}^{(0)} \end{pmatrix} . \tag{3.11}$$

Let us look more closely at the structure of each block in the above matrix. First of all, by (A.2)

$$\left(N_{--}\right)_{ij} \sim a^{n/2-j} f_{ij}^{(N)}(\tilde{b}_l) , \qquad \left(M_{--}\right)_{ij} \sim a^{n/2-j} f_{ij}^{(M)}(\tilde{b}_l) , \tag{3.12}$$

which implies

$$\left(N_{--}\right)_{ij}^{-1} \left(M_{--}\right)_{jk} = f_{ij}^{(N)-1}(\tilde{b}_l) f_{jk}^{(M)}(\tilde{b}_l) = \Pi_{ik}^{-}(\tilde{b}_l) . \tag{3.13}$$

Furthermore, since $N_{--}^{-1}$ vanishes in the limit $a \to 0$, we find that

$$N_{--}^{-1} M_{-+}^{(0)} + \mathcal{N} M_{++}^{(0)} = N_{--}^{-1} M_{-+}^{(0)} - N_{--}^{-1} N_{-+}^{(0)} \left(N_{++}^{(0)}\right)^{-1} M_{++}^{(0)} \to 0 . \tag{3.14}$$

Therefore, the period matrix has the following block-diagonal form in the double-scaling limit

$$\Pi \longrightarrow \begin{pmatrix} \Pi^{-} & 0 \\ 0 & \Pi^{+} \end{pmatrix} . \tag{3.15}$$

The upper block $\Pi^-$ is actually the period matrix of the near-critical spectral curve $\Sigma_-$ (2.9) since the cycles $\{\tilde{A}_i, \tilde{B}_i\}$, for $i \leq [n/2]$ form a standard homology basis for $\Sigma_-$. Similarly, the lower block $\Pi^+$ is the period matrix of $\Sigma_+$. So in the limit $a \to 0$ the curve $\Sigma$ factorizes as $\Sigma_- \cup \Sigma_+$. The fact that the period matrix factorizes is evidence of the more stringent claim that the whole theory consists of two decoupled sectors $\mathcal{H}_-$ and $\mathcal{H}_+$ in the double-scaling limit. Note that although we did not consider it, the U(1) associated to $S$ only couples to the $\mathcal{H}_+$ sector.

We can extend this discussion to include other $F$-terms that are derived from the glueball superpotential. For example, consider the 3-point vertex

$$V_{ijk}^{(3)} = \frac{\partial^3 F_0}{\partial \tilde{S}_i \partial \tilde{S}_j \partial \tilde{S}_k} . \tag{3.16}$$

Here, the $\tilde{S}_i$ as defined as in (1.11) but with respect to the cycles $\tilde{A}_i$. They are related to the $S_i$ by an electro-magnetic duality transformation. There is a closed expression for these couplings of the form [36–39]

$$V_{ijk}^{(3)} = \frac{1}{N\varepsilon} \sum_{l=1}^{2s} \mathrm{Res}_{b_l} \frac{\omega_i\, \omega_j\, \omega_k}{dx dy} , \tag{3.17}$$

where $\{\omega_j\}$ are the holomorphic 1-forms normalized with respect to the basis $\{\tilde{A}_i, \tilde{B}_i\}$. So we can deduce the behaviour of the couplings from our knowledge of the scaling of $\omega_j$. This is derived in the appendix. We find that the couplings are regular as $a \to 0$, except if $i, j, k \leq [n/2]$ in which case,

$$V_{ijk}^{(3)} \longrightarrow \left(N\varepsilon a^{m+n/2+1}\right)^{-1} \sum_{l=1}^{n} \mathrm{Res}_{\tilde{\sigma}_l} \frac{\tilde{\omega}_i\, \tilde{\omega}_j\, \tilde{\omega}_k}{d\tilde{x} dy_-} , \tag{3.18}$$

where the $\{\tilde{\omega}_i\}$ are the one-forms on $\Sigma_-$. Therefore, in the double-scaling limit proposed in (2.12), we find that these interactions remain finite $\sim \Delta^{-1}$, while the other 3-point vertices $\to 0$. This is yet further evidence of the decoupling of the Hilbert space into two decoupled sectors where the interactions in the $\mathcal{H}_-$ sector remain finite in the double-scaling limit while those in $\mathcal{H}_+$ go to zero. Notice, also that these interactions of the $\mathcal{H}_-$ sector depend universally on $\Sigma_-$.

The final $F$-term quantity that we consider is the Hessian matrix for the glueball superfields

$$H^{(2)}_{jk} = \frac{\partial^2 W_{\rm gb}}{\partial \tilde{S}_j \partial \tilde{S}_k} \ . \tag{3.19}$$

Using (3.17) we find

$$H^{(2)}_{jk} = \sum_{i=1}^{s} N_i \frac{\partial^3 F_0}{\partial \tilde{S}_i \partial \tilde{S}_j \partial \tilde{S}_k} = \frac{1}{N\varepsilon} \sum_{l=1}^{2s} \text{Res}_{b_l} \frac{T\, \omega_j\, \omega_k}{dx dy} \ , \tag{3.20}$$

where we have defined the 1-form $T$

$$T = N\varepsilon \sum_{i=1}^{s} \frac{\partial y\, dx}{\partial S_i} \ . \tag{3.21}$$

It is known that $T$ can be can be written simply in terms of the on-shell Seiberg-Witten curve [40]:

$$T = d\log(P_N + y_{\rm SW}) \ . \tag{3.22}$$

In the limit $a \to 0$, we can take the near-critical expressions for $y_{\rm SW}$ in (2.28) and for $P_N(x) = 2\Lambda^N$ to get the behaviour

$$T \longrightarrow \Lambda^{-r-n/2} H_r(0) \frac{N}{n+2r} a^{n/2} d\sqrt{\tilde{B}(\tilde{x})} \sim N a^{n/2} \ . \tag{3.23}$$

We also need

$$dy \longrightarrow a^{m+n/2} d\big(\tilde{Z}_m(x)\sqrt{\tilde{B}(\tilde{x})}\big) \sim a^{m+n/2} \ . \tag{3.24}$$

The scaling of the holomorphic differentials is determined in the appendix.

Counting the powers of $N$ and $a$, we find that for any $j$ and $k$, $H^{(2)}_{jk}$ goes like an inverse power of $a$ and hence diverges in the double-scaling limit (the powers of $N$ cancel). This, however, presents us with a puzzle. In the case without double points described in [4], the Hessian was shown to vanish for the $\mathcal{H}_-$ sector, i.e. $j,k \leq [n/2]$. Let us see how this is compatible with the scaling we have just seen. In the case, $j,k \leq [n/2]$,

$$H^{(2)}_{jk} \sim a^{-(m+1)} \sum_{l=1}^{n} \text{Res}_{\tilde{b}_l} \left[ \frac{d\sqrt{\tilde{B}_n(\tilde{x})}\ \tilde{\omega}_j\, \tilde{\omega}_k}{d\tilde{x}\ d\Big(\tilde{Z}_m(\tilde{x})\sqrt{\tilde{B}_n(\tilde{x})}\Big)} \right] , \tag{3.25}$$

where

$$\tilde{\omega}_j = \frac{\tilde{L}_j(\tilde{x})}{\sqrt{\tilde{B}_n(\tilde{x})}} d\tilde{x}. \tag{3.26}$$

and $\tilde{L}_j(\tilde{x})$ is a polynomial of degree $[n/2]-1$. Note that the differential $\tilde{\omega}_j\,\tilde{\omega}_k/d\tilde{x}$ has simple poles at $\tilde{x}=\tilde{b}_l$ on the curve $\Sigma_-$:

$$\frac{\tilde{\omega}_j\,\tilde{\omega}_k}{d\tilde{x}} = \frac{\tilde{L}_j(\tilde{x})\,\tilde{L}_k(\tilde{x})}{\tilde{B}_n(\tilde{x})}d\tilde{x}\,, \tag{3.27}$$

but has no pole at $\tilde{x}=\infty$. For example for $n$ odd, we find

$$\frac{\tilde{\omega}_j\,\tilde{\omega}_k}{d\tilde{x}} \longrightarrow \frac{d\tilde{x}}{\tilde{x}^3}\ . \tag{3.28}$$

This means that in the case with no double points, $m=0$, the Hessian matrix elements (3.25) vanish identically:

$$H^{(2)}_{jk} \sim a^{-1}\sum_{l=1}^{n}\mathrm{Res}_{\tilde{b}_l}\,\left[\frac{\tilde{\omega}_j\,\tilde{\omega}_k}{d\tilde{x}}\right] = 0\,, \tag{3.29}$$

because the sum of all residues of a meromorphic differential on the compact near-critical curve $\Sigma_-$ is identically zero. This is precisely the result found in [4]. On the other hand, if $m>0$, the Hessian matrix element will not vanish in general, because the differential on the right-hand side of (3.25) has extra simple poles at the roots of

$$2\tilde{Z}_m'(\tilde{x})\tilde{B}_n(\tilde{x}) + \tilde{Z}_m(\tilde{x})\tilde{B}_n'(\tilde{x}) = 0\ . \tag{3.30}$$

This result is very significant because it highlights an important difference between the case with and without double points. Even though we do not have control over the kinetic terms of the glueball states, we take this behaviour of the Hessian matrix to signal that, with double points, the masses of the glueball fields become very large in the double-scaling limit. This is to be contrasted with the case without double points studied in [4], where the appearance of the $[n/2]$ massless glueballs was interpreted as evidence that supersymmetry is enhanced to $\mathcal{N}=2$ in the double-scaling limit.

## 4. The double-scaling limit of the free energy

In this section, we will consider the behaviour of the free energy in the limit $a\to 0$. The most powerful methods for calculating the $F_g$ involves using orthogonal polynomials which we review in section 4.2; however, as we shall see, these techniques appear only to be successful for the cases with one or two branch points and any additional number of double points. The only known way to calculate the $F_g$ in general involves analyzing the loop equations and in particular using the algorithms recently developed in [41, 42]. We use these techniques to prove our scaling ansatz (2.12) in section 4.3. Before this, it is useful to consider a situation which is exactly solvable. This is the case when there are two branch points and no double points: $n=2$ and $m=0$.



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### 4.1 Simple example: two branch points

In this section, we consider in some detail the case when the critical point involves the collision of two branch points. Most of the formulae that we use are taken from [43]. With two branch points and no double points, we can engineer as in section 2 with a curve of the form

$$y^2 = (x^2 - a^2)(x^2 - b^2) \tag{4.1}$$

in the limit $a \to 0$. In order to be on-shell, we require

$$b^2 = a^2 + 4\Lambda^2 \tag{4.2}$$

because then

$$y^2 = (x^2 - a^2 - 2\Lambda^2)^2 - 4\Lambda^4 \,, \tag{4.3}$$

which is the Seiberg-Witten curve of an SU(2) theory. The potential required is

$$W(x) = N\varepsilon\left(\frac{1}{3}x^3 - (a^2 + 2\Lambda^2)x\right) \, . \tag{4.4}$$

As $a \to 0$, by hypothesis, there is a light dibaryon with mass controlled by

$$\begin{aligned} \tilde{S} &= N \int_{-a}^{a} \sqrt{(x^2 - a^2)(x^2 - a^2 - 2\Lambda^2)}\, dx \\ &= \frac{2}{3} N \sqrt{a^2 + 2\Lambda^2}\big(2(a^2 + \Lambda^2)E(a^2/(a^2 + 2\Lambda^2)) + 2\Lambda^2 K(a^2/(a^2 + 2\Lambda^2))\big) \, . \end{aligned} \tag{4.5}$$

At leading order for small $a$

$$\tilde{S} = \frac{\pi}{\sqrt{2}} N\Lambda a^2 + \cdots \tag{4.6}$$

Notice that the double-scaling limit involves $N \to \infty$ with $Na^2$ fixed, in agreement with (2.11). One way to “see” the dibaryon is to calculate the low energy couplings of the two U(1)’s. The coupling constant matrix has the form

$$\tau \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{4.7}$$

where $\tau$ is the period matrix of the curve which can be written explicitly in terms of elliptic functions as [43]

$$\tau = \frac{K(a^2/(a^2 + 2\Lambda^2))}{E(2\Lambda^2/(a^2 + 2\Lambda^2))} \, . \tag{4.8}$$

In the limit $a \to 0$, the leading-order behaviour is

$$\tau = \frac{\pi}{\log(\alpha/a^2)} \, , \tag{4.9}$$

for some constant $\alpha$. This kind of logarithmic running of the coupling is characteristic of a one-loop effect of a particle of mass $\sim \tilde{S}$ which is magnetically charged under the two U(1)’s.

However, in this case we can also extract the behaviour of the higher genus $F_g$ as $a \to 0$. In the present case, near the critical point we can approximate the curve by

$$y^2 \longrightarrow 4\Lambda^2(x^2 - a^2) \ . \tag{4.10}$$

Now we can appeal to universality in the double-scaling limit. The reduced curve above is precisely the one encountered in the matrix model with a Gaussian potential

$$W(x) = N\varepsilon\Lambda x^2 \ . \tag{4.11}$$

So the free energy of our original matrix model in the double-scaling limit should be equal to the free energy of the Gaussian model with the 't Hooft coupling $S = g_s\hat{N}$ of the latter identified with $\tilde{S}$ above. The Gaussian matrix model can be solved exactly [25] yielding

$$F_{\text{Gaussian}}(S) = \frac{1}{2}g_s^{-2}S^2\log S - \frac{1}{12}\log S - \frac{1}{240}g_s^2 S^{-2} + \sum_{g>2}\frac{B_g}{2g(2g-2)}g_s^{2g-2}S^{2-2g} \ . \tag{4.12}$$

Hence, by appealing to universality, we deduce that in the double-scaling limit of the original theory as $a \to 0$

$$F_g \sim \tilde{S}^{2-2g} \propto |Na^2|^{2-2g} \,, \tag{4.13}$$

which verifies in a simple example the scaling hypothesis (2.11). In particular, notice that the logarithm in $F_0$ is directly attributed to a one-loop renormalization of the coupling due to the light state.

In [44], the methods of [41, 42, 65] are used to evaluate the genus one and two terms of the matrix model free energy for the class of double-scaling limits considered in [4]. The simplest case corresponds to the above conifold singularity with two branch points colliding and no double points, $n = 2, m = 0$, and the results match the genus zero, one and two terms in (4.12). This is precisely due to the fact that the near-critical spectral curve reduces to the curve of a Gaussian matrix model, which is simply a Riemann sphere.

### 4.2 Orthogonal polynomials

Motivated by the study of random surfaces and 2-d quantum gravity, the double scaling limits of multi-cut matrix models were investigated in the early 90's using orthogonal polynomials. This technique was originally developed in [45] as an alternative way to calculate the matrix model free energy

$$\exp(F) = \int d\hat{\Phi}\, \exp\Big(-g_s^{-1}\mathrm{Tr}\, W(\hat{\Phi})\Big) = \int \prod_{i=1}^{\hat{N}} dx_i\, \Delta^2(x)\, \mathrm{e}^{\,-g_s^{-1}\sum_i W(x_i)} \,, \tag{4.14}$$

where $\Delta(x) = \prod_{i\neq j}(x_i - x_j)$ is the Vandermonde determinant, and its main features can be found in the reviews [46, 47]. The basic idea is to calculate the integral in (4.14) by introducing the set of polynomials $P_n(x)$ orthogonal with respect to the measure $\int dx\, \mathrm{e}^{\,-g_s^{-1}W(x)}\, P_n(x)\, P_m(x) = h_n\delta_{mn}$, where $P_n(x)$ is a polynomial of degree $n$ normalized such that $P_n(x) = x^n + \cdots$. Orthogonal polynomials satisfy recursion relations of the form $xP_n(x) = P_{n+1}(x) + \sigma_n P_n(x) + r_n P_{n-1}(x)$, where $r_n$ and $\sigma_n$ are $x$-independent purely numerical coefficients, and $\sigma_n = 0$ if the potential is even. They can be determined by solving the non-linear recursive equations

$$n\, g_s\, h_{n-1} = \int \mathrm{e}^{\,-g_s^{-1} W(x)}\, W'(x)\, P_n(x)\, P_{n-1}(x)\,, \tag{4.15}$$

$$0 = \int \mathrm{e}^{\,-g_s^{-1} W(x)}\, W'(x)\, P_n(x)\, P_n(x)\,. \tag{4.16}$$

In the large-$\hat{N}$ limit, the re-scaled index $n/\hat{N}$ becomes a continuous variable $\xi \in [0,1]$, and the orthogonal polynomial method becomes useful provided that we have an appropriate ansatz for the large-$n$ behaviour of $r_n$ and $\sigma_n$. In the double scaling limit, the method provides important information about the resulting models. The reason is that the free energy turns out to be described in terms of particular solutions of specific classical integrable hierarchies. The relevant solutions are singled out by additional equations known as "string equations". This important result permits the calculation of the complete perturbative (topological) expansion of the free energy in terms of the double scaled parameters, and it also provides non-perturbative information about the resulting models. Moreover, the relationship with integrable hierarchies makes it possible to write flow equations that interpolate between different models.

The simplest large $n$ ansatz is $r_n \to r(\xi; S)$ and $\sigma_n \to \sigma(\xi; S)$, such that both coefficients become smooth functions of $\xi$ and the 't Hooft coupling $S = g_s \hat{N}$. It corresponds to saddle point configurations whose large-$\hat{N}$ resolvent has just one cut [45, 48]. In this case, to describe all the resulting double scaling limits, it is enough to restrict ourselves to the case of Hermitian matrix models with even potentials. Then, the topological large-$\hat{N}$ expansion of the free energy exhibits a singular behaviour of the form

$$F = \sum_{g\geq 0} F_g(S)\, \hat{N}^{2-2g}\,, \qquad F_g(S) \sim (S - S_c)^{(2+\frac{1}{m+1})(1-g)}\,, \quad m \geq 1\,, \tag{4.17}$$

up to a few regular terms in $F_0$. In this approach it is customary to keep the potential $W$ fixed and consider just the dependence on $S$; however, we could equivalently fix the value of $S = S_c$ and look at the critical behaviour as a function of the coupling constants in $W$.[9] Adopting the first point of view, both $S_c$ and the integer $m$ depend on $W$. In the saddle point method, the critical behaviour (4.17) corresponds to the case when $m$ double points come together with one of the branch points of the resolvent, which can be realized with potentials of degree larger or equal $2(m+1)$. It leads to the double scaling limit $\hat{N} \to \infty$, $S \to S_c$, with $\hat{N}(S - S_c)^{1+\frac{1}{2(m+1)}} =$ finite. Using orthogonal polynomials, this limit can be performed with a scaling ansatz of the form

$$r(\xi; S) = r_c \left( 1 + \frac{u(z)}{\hat{N}^{\frac{2}{2m+3}}} + \cdots \right)\,, \qquad \sigma(\xi; S) = 0\,, \tag{4.18}$$

together with $n g_s = \xi S = S_c \left( 1 - z\, \hat{N}^{-\frac{2m+2}{2m+3}} \right)$. It is worth noticing that, in the saddle point approximation, this ansatz corresponds to a resolvent with two branch points symmetrically

[9] This is exactly what it done in the Dijkgraaf-Vafa matrix model, where $S$ is fixed by the requirement to extremize the glueball superpotential.

located at [45, 48]

$$x = \pm x_b = \pm\sqrt{r(1;S)} \sim \pm\big(x_c + O(\hat{N}^{-\frac{2}{2m+3}})\big)\,. \tag{4.19}$$

Then,

$$a = x_b - x_c \sim \big(1 - S/S_c\big)^{\frac{1}{m+1}} = \hat{N}^{-\frac{2}{2m+3}}\ z^{\frac{1}{m+1}}\,, \tag{4.20}$$

and the double scaling limit prescription can be written as $\hat{N}a^{m+\frac{3}{2}} = \text{const.}$, which is in agreement with eq. (2.12), and ensures that the free energy scales as in (2.11) with $n = 1$. The resulting models are well known [46]. The free energy is given by $d^2F/dz^2 = u$, where $u = u(z)$ is a particular solution of the KdV hierarchy that satisfies a string equation depending on $m$. In the simplest case, $m = 1$, the string equation is $z = u^2 + u''/3$, which is known as Painlevé I; the resulting model describes pure 2-d gravity. For $m > 1$ the model corresponds to the $(2m+1, 2)$ CFT minimal model coupled to 2-d gravity. It can be shown that the $m$-th string equation is associated to the $m+1$-th flow of the KdV hierarchy. Although in the matrix model approach $m \geq 1$, the integrable hierarchy formulation allows one to construct an additional model associated to the $1^{\text{st}}$ flow of the hierarchy. Actually it corresponds to a topological theory that underlies all the other double scaled models, and it was identified with 2-d topological gravity [49].

Saddle point configurations with two cuts are recovered with a "period-two" ansatz; i.e., by assuming that $r_{2n}$, $r_{2n+1}$, $\sigma_{2n}$ and $\sigma_{2n+1}$ approach different smooth functions in the large-$n$ limit [50]. However, the resulting cuts always open up around minima of the potential $W(x)$ [48, 51] and it is not known how to describe saddle point configurations with cuts open around the maxima of $W(x)$ using orthogonal polynomials. This includes, for instance, the two-cut phase of the matrix model specified by $W = \frac{1}{3}g_3x^3 + g_1x$, which was extensively discussed in [43, 52], as well as in section 4.1. Actually, this limitation poses a serious restriction for the use of the orthogonal polynomial method in the context of (1.10). In any case, it is interesting to review the main results achieved by considering the class of two-cut matrix models that can be studied using this method.

Two-cut Hermitian matrix models with even potentials were investigated in [53]. They exhibit critical behaviours corresponding to cases where two branch points come together with an odd number of double points, and lead to the same double scaling limits that were previously found in the study of unitary matrix models [54], which are described in terms of the modified KdV hierarchy. The interpretation of the resulting models in terms of super CFTs coupled to 2-d supergravity was suggested in the early 90's [53, 55], and it has been recently clarified and supported by Klebanov, Maldacena and Seiberg [56]. However, these are not the only two-cut models that have been studied using orthogonal polynomials. Two-cut Hermitian matrix models with both even and odd terms in the potential were considered in [55, 57, 58]. Remarkably, the corresponding double scaling limits are described in terms of several integrable hierarchies associated to $sl(2,\mathbb{C})$. Following [58], the results can be summarized as follows. First, consider the case of real potentials. In the saddle point approximation, their critical behaviours correspond to configurations where an odd number of double points come together with two branch points [55, 58]. Their double scaling limits are described in terms of solutions of the non-linear Schrödinger (NLS) hierarchy singled out by string equations associated only to the "even" flows of the hierarchy.

However, as pointed out in [58], the most general critical behaviour, where an arbitrary number of double points collide with two branch points, is obtained by considering complex potentials. Namely, the potential has to be taken such that $W^*(x) = W(-x)$, which makes the coefficients $\sigma_n$ pure imaginary.[10] This gives rise to double scaling limits described in terms of solutions of the Zakharov-Shabat (ZS) hierarchy specified by string equations associated to the complete set of flows, but the first one. The resulting models are described by two real functions $f(z)$ and $g(z)$. They enter the ansatz for the orthogonal polynomial coefficients, which is of the form

$$r_n = r_c \left( 1 + (-1)^n \frac{f(z)}{\hat{N}^{\frac{1}{m+2}}} + \cdots \right) , \quad \sigma_n = i\sigma_c \left( b + (-1)^n \frac{g(z)}{\hat{N}^{\frac{1}{m+2}}} + \cdots \right) , \tag{4.21}$$

together with $\xi S = S_c \left(1 - z\, \hat{N}^{-\frac{m+1}{m+2}}\right)$. Here, $b$ is an arbitrary number, and $m$ is the number of double points. In the saddle point approximation, this ansatz gives rise to a resolvent where the two colliding branch points are located at [48, 51]

$$\begin{aligned} x = x_b^{\pm} &= \left. \frac{\sigma_{2n} + \sigma_{2n+1} \pm \sqrt{\left(\sigma_{2n} - \sigma_{2n+1}\right)^2 - 4\left(\sqrt{r_{2n}} - \sqrt{r_{2n+1}}\right)^2}}{2} \right|_{\xi=1} \\ &\sim ib\sigma_c \pm O(\hat{N}^{-\frac{1}{m+2}}) \, . \end{aligned} \tag{4.22}$$

Thus, in this case,

$$a = x_b^+ - x_b^- \sim \left(1 - S/S_c\right)^{\frac{1}{m+1}} = \hat{N}^{-\frac{1}{m+2}} \, z^{\frac{1}{m+1}} \, , \tag{4.23}$$

and the double scaling limit prescription can be written as $\hat{N} a^{m+2} = \text{const.}$, which is again in agreement with eq. (2.12) for $n = 2$. The free energy is given by $d^2F/d^2z = \left(f^2(z) - g^2(z)\right)/4$. For $m = 1$ the string equations are

$$zf + f(g^2 - f^2) + 2f'' = 0 \, , \qquad zg + g(g^2 - f^2) + 2g'' = 0 \, , \tag{4.24}$$

and for $m = 2$ they read

$$2zf + 3g'(g^2 - f^2) + 2g''' = 0 \, , \qquad 2zg + 3f'(g^2 - f^2) + 2f''' = 0 \, . \tag{4.25}$$

They exhibit that the string equations corresponding to odd $m$ admit the reduction $g = 0$ and the analytical continuation $g \to ig$, which lead to the mKdV and the NLS hierarchies, respectively. Other reductions, as well as the relationship between the partition function and the tau-functions of the hierarchies, have been discussed in [58]. It is straightforward to check that the free energy scales exactly as in (2.11) with $n = 2$.

An interesting observation about the structure of these two-cut matrix models was made in [55]. There, it was shown that the formulation in terms of the ZS hierarchy allows one to define a model associated to the 1st flow which is of topological nature. This is

[10] This is equivalent to consider anti-Hermitian matrix models, whose eigenvalues are purely imaginary, with real potentials.

similar to what happens in the one-cut case, where the $1^{\rm st}$ flow of the KdV hierarchy leads to 2-d topological gravity. By comparison, it is natural to expect that the topological model of [55] underlies the structure of all the ZS-related two-cut matrix models, although the proper interpretation of this topological phase is still unclear.

Another important feature pointed out in [51] and, mostly, in [56] is the fact that the string equations of these models admit an integration constant. For the models associated to the ZS hierarchy, this is a consequence of the invariance under the $\mathrm{SO}(1,1)$ transformations $f \pm g \rightarrow \mathrm{e}^{\pm\beta}(f \pm g)$. This constant shows up in the perturbative expansion of the free energy, although the leading order term remains independent of it. The only exception is the topological model of [55], where the free energy is proportional to a non-vanishing integration constant. However, in [55] the presence of this constant in the non-topological models was missed, mainly because it was assumed that the potential is real and, consequently, the colliding double points and branch points are on the real axis. In [56], double points were allowed to be complex, the integration constant was understood in terms of the period of the hyper-elliptic $y = y(x)$ curve of the matrix model around the cut joining the two colliding branch points, and its physical interpretation was clarified. When $g \rightarrow ig$, and the ZS hierarchy changes to the NLS hierarchy, the constant becomes complex too, and it can be related to the difference between the number of eigenvalues sitting on the two cuts [51, 56].

Motivated by the correspondence between the period-2 large-$n$ ansatz and two-cut saddle point configurations, a study of multi-cut configurations by considering more general period-$q$ large-$n$ behaviours; i.e., $r_{lq+i} \rightarrow r_i(\xi; S)$ and $\sigma_{lq+i} \rightarrow \sigma_i(\xi; S)$, for $i = 0, 1, \ldots, q-1$, was initiated. However, for $q \geq 3$ this sort of ansatz is only able to reproduce multi-cut saddle point configurations for very special types of potentials. For instance, for $q = 3$ and even potentials of degree six with three minima, the correspondence turns out to works only in the particular case when all the minima are degenerate [48]. In other words, the appropriate large-$n$ ansatz corresponding to generic saddle point configurations with three or more cuts in not known, which prevents the use of the orthogonal polynomial method to investigate the properties of these configurations and their double scaling limits. Several aspects of this problem were discussed in [59].

An interesting particular set of multi-cut matrix models partially studied using orthogonal polynomials are provided by the, so-called, "orbifold (Hermitian) matrix models" proposed in [53]. They give rise to saddle-point configurations with $2k$ cuts symmetrically radiating from the origin. The critical behaviours considered in that paper correspond to cases where the end points of all the cuts simultaneously collide together with $2km - 1$ double points, for $m \geq 1$. The resulting double-scale models are expected to exhibit a $\mathbf{Z}_k$ symmetry. The simplest potential that gives rise to such configuration is $W(x) = x^8 - x^4$ for a matrix model defined over matrices $\hat{\Phi}$ such that $\hat{\Phi}^2$ is hermitian, for which there are four branch points and three double points in the critical region. This model was studied in [53] by means of a very symmetrical period-4 large-$n$ ansatz for the coefficients $r_n$. However, both its physical interpretation and the associated integrable hierarchy remain mysterious still. Nevertheless, it can verified from the resulting generalized string equations, that the free energy scales as in (2.11) with $n = 2k$ and $m = 2km - 1$.

Apart from the aforementioned $\mathbf{Z}_k$ models, the fact that the orthogonal polynomial method seems to work for one or two branch cuts may be related to the fact that resulting near-critical curves are both of genus zero. In fact, this is related to the fact that the string equations in both cases can be formulated in terms of the Virasoro algebra of a chiral boson on the complex plane. For the case of one branch point there is a branch cut extending to infinity and the boson is twisted, $\phi(ze^{2\pi i}) = -\phi(z)$.

### 4.3 The loop equations

In this section, consider a radically different approach for investigating the free energy which involves a recursion relation for the $F_g$. Before we come to the loop equation itself we need to define the $p$-loop correlator, or $p$-point loop function, as

$$\begin{aligned} W(x_1, \ldots, x_p) &\equiv \hat{N}^{p-2} \left\langle \mathrm{Tr}\frac{1}{x_1 - \hat{\Phi}} \cdots \mathrm{Tr}\frac{1}{x_p - \hat{\Phi}} \right\rangle_{\mathrm{conn}} \\ &= \frac{d}{dV}(x_p) \cdots \frac{d}{dV}(x_1) F \,, \qquad p \geq 2 \,. \end{aligned} \tag{4.26}$$

It has the following genus expansion

$$W(x_1, \ldots, x_p) = \sum_{g=0}^{\infty} \frac{1}{\hat{N}^{2g}} W^{(g)}(x_1, \ldots, x_p) \,. \tag{4.27}$$

In [41], Eynard found a solution to the matrix model loop equations that allows to write down an expression for these multi-loop correlators at any given genus in terms of a special set of Feynman diagrams. The various quantities involved depend only on the spectral curve of the matrix model and in particular one needs to evaluate residues of certain differentials at the branch points of the spectral curve.

This algorithm and its extension to calculate higher genus terms of the matrix model free energy [42] represent major progress in the solution of the matrix model via loop equations [60 – 63, 52, 64, 65]. This is particularly important because,as we have seen in the last section, the orthogonal polynomial approach can only applied in very special cases. A nice feature of the loop equation approach is that they show directly how the information is encoded in the spectral curve. In particular, we will be able to make some precise statements on the double-scaling limits of higher genus quantities simply by studying the double-scaling limit of the spectral curve and its various differentials.

Given the matrix model spectral curve for an $s$-cut solution in the form (2.4) the genus zero 2-loop function is given by

$$\begin{aligned} W(x_1, x_2) = &-\frac{1}{2(x_1 - x_2)^2} + \frac{\sqrt{\sigma(x_1)}}{2\sqrt{\sigma(x_2)}(x_1 - x_2)^2} \\ &- \frac{\sigma'(x_1)}{4(x_1 - x_2)\sqrt{\sigma(x_1)}\sqrt{\sigma(x_2)}} + \frac{A(x_1, x_2)}{4\sqrt{\sigma(x_1)}\sqrt{\sigma(x_2)}} \,, \end{aligned} \tag{4.28}$$

where $A$ is a symmetric polynomial given by

$$A(x_1, x_2) = \sum_{i=1}^{2s} \frac{\mathcal{L}_i(x_2)\sigma(x_1)}{x_1 - \sigma_i} \,, \tag{4.29}$$

with

$$\mathcal{L}_i(x_2) = \sum_{l=0}^{s-2} \mathcal{L}_{i,l} x_2^l = -\sum_{j=1}^{s-1} L_j(x_2) \int_{A_j} \frac{dx}{\sqrt{\sigma(x)}} \frac{1}{(x-\sigma_i)} \tag{4.30}$$

and $s$ is the number of cuts. The polynomials $L_j(x)$ are related to the holomorphic 1-forms and defined in the appendix.

The genus zero 2-loop function for coincident arguments is

$$\begin{aligned} W(x_1,x_1) &= \lim_{x_2\to x_1} W(x_1,x_2) = -\frac{\sigma''(x_1)}{8\sigma(x_1)} + \frac{\sigma'(x_1)^2}{16\sigma(x_1)^2} + \frac{A(x_1,x_1)}{4\sigma(x_1)} \\ &= \sum_{i=1}^{2s} \frac{1}{16(x-\sigma_i)^2} - \frac{\sigma_i''}{16\sigma_i'(x-\sigma_i)} + \frac{\mathcal{L}_i(x)}{4(x-\sigma_i)} \ . \end{aligned} \tag{4.31}$$

The other important object is the differential

$$\begin{aligned} &dS_{2i-1}(x_1,x_2) = dS_{2i}(x_1,x_2) \\ &= \frac{\sqrt{\sigma(x_2)}}{\sqrt{\sigma(x_1)}} \left( \frac{1}{x_1-x_2} - \frac{L_i(x_1)}{\sqrt{\sigma(x_2)}} - \sum_{j=1}^{s-1} C_j(x_2) L_j(x_1) \right) dx_1 \ , \end{aligned} \tag{4.32}$$

where $i = 1, \ldots, s$ and

$$C_j(x_2) = \int_{A_j} \frac{dx}{\sqrt{\sigma(x)}} \frac{1}{(x-x_2)} \ . \tag{4.33}$$

A crucial aspect of the one-form (4.32) is that it is analytic in $x_2$ in the limit $x_2 \to \sigma_{2i-1}$ or $\sigma_{2i}$ [41]

$$\lim_{x_2\to\sigma_i} \frac{dS_i(x_1,x_2)}{\sqrt{\sigma(x_2)}} = \frac{1}{\sqrt{\sigma(x_1)}} \left( \frac{1}{x_1-x_2} - \sum_{j=1}^{s-1} L_j(x_1) \int_{A_j} \frac{dx}{\sqrt{\sigma(x)}} \frac{1}{(x-x_2)} \right) dx_1 \ . \tag{4.34}$$

The subtlety is that in the definition of (4.33), the point $x_2$ is taken to be outside the loop surrounding the $j$-th cut, whereas in (4.34), $x_2$ is inside the contour. Note also that

$$A(x_1,x_2) = -\sum_{i=1}^{2s} \left( \sum_{j=1}^{s-1} L_j(x_2) C_j(\sigma_i) \right) \frac{\sigma(x_1)}{x_1-\sigma_i} \tag{4.35}$$

and in particular

$$A(x_1,\sigma_i) = \mathcal{L}_i(x_1)\sigma'(\sigma_i) \ . \tag{4.36}$$

The expression of $W^{(g)}(x_1,\ldots,x_p)$ can be found by evaluating a series of Feynman diagrams of a cubic field theory on the spectral curve [41]. To this end, define the set $\mathcal{T}_p^{(g)}$ of all possible graphs with $n$ external legs and with $g$ loops. They can be described as follows: draw all *rooted* skeleton trees ( trees whose vertices have valence $1, 2$ or $3$ ), with $p+2g-2$ edges. Draw arrows on the edges from the root towards the leaves. Then draw in all possible ways $p-1$ external legs and $g$ inner edges with the constraint that all the vertices of the whole graph have valence three, namely that are always three and only three edges emanating from any given vertex. Each such graph will also have some symmetry factor [41].

Each diagram in then weighted in the following way. To each arrowed edge that is part of the skeleton tree going from a vertex labelled by $x_1$ to a vertex labelled by $x_2$ associate the differential $dS(x_1, x_2)$ (4.32). To each non-arrowed edge associate a genus zero 2-loop differential $G(x_1, x_2) = W(x_1, x_2)\, dx_1\, dx_2$ and to each *internal* vertex labelled by $x_1$ associate the factor $(2\varepsilon N y(x_1) dx_1)^{-1}$. For any given tree $T \in \mathcal{T}_p^{(g)}$, with root $x_1$ and leaves $x_j, j = 2, \ldots, p$ and with $p + 2g - 2$ vertices labelled by $x'_v, v = 1, \ldots, p + 2g - 2$, so that its inner edges are of the form $v_1 \to v_2$ and its outer edges are of the form $v \to j$, we define the weight of the graph as follows

$$\begin{aligned} \mathcal{W}(T) = & \frac{1}{(\varepsilon N)^{p+2g-2}} \prod_{v=1}^{p+2g-2} \sum_{i_v=1}^{2s} \mathrm{Res}_{x'_v \to b_{i_v}} \frac{1}{2y(x'_v)dx'_v} \prod_{\text{inner edges } v \to w} dS_{i_v}(x'_v, x'_w) \\ & \times \prod_{\text{inner non-arrowed edges } v' \to w'} G_2(x'_{v'}, x'_{w'}) \prod_{\text{outer edges } v \to j} G_2(x'_v, x_j) \end{aligned} \tag{4.37}$$

In order to find an expression for $F_g$, $g > 1$, one should consider the same graphs relevant for $W^{(g)}(x_1)$ and do then the following [42]:
**(i)** Eliminate the first arrowed edge of the skeleton tree. Labelling the first vertex by $x_1$ and the second vertex by $x_2$, this amounts to dropping the factor $dS(x_1, x_2)$.

**(ii)** The factor $(2\varepsilon N y(x_2) dx_2)^{-1}$ has to be dropped and replaced by

$$\frac{\int_{q_0}^{x_2} y(s) ds}{y(x_2) dx_2} . \tag{4.38}$$

Note that when evaluating the final residues at $x_2 = \sigma_i$, one needs to expand the above integral by setting $q_0 = \sigma_i$ [42]. It is also understood that the evaluation of the residues starts from the outer branches and proceeds towards the root. This procedure does not apply for the genus one free energy whose expression has in any case been found via the loop equations in [52, 64, 65].

We will consider the $a \to 0$ limit of each element in (4.37). Using the results in the appendix for the scaling of $L$, it is straightforward to argue that for a branch point $b_i$ in the critical region

$$\begin{aligned} & dS_i(x_1, x_2) \longrightarrow d\tilde{S}_i(\tilde{x}_1, \tilde{x}_2) \\ & = \frac{\sqrt{\tilde{B}(\tilde{x}_2)}}{\sqrt{\tilde{B}(\tilde{x}_1)}} \left( \frac{1}{\tilde{x}_1 - \tilde{x}_2} - \frac{\tilde{L}_i(\tilde{x}_1)}{\sqrt{\tilde{B}(\tilde{x}_2)}} - \sum_{j=1}^{p} \tilde{C}_j(\tilde{x}_2) \tilde{L}_j(\tilde{x}_1) \right) d\tilde{x}_1 , \end{aligned} \tag{4.39}$$

where $d\tilde{S}_i$ is the analogous differential on $\Sigma_-$ and $L_j(x) \to a^{n/2-1} \tilde{L}_j(\tilde{x})$ for $j \leq [n/2]$. Conversely, the differentials $dS_i(x_1, x_2)$ where $i$ labels a branch point of the spectral curve that remains outside of the critical region give a vanishing contribution in the double-scaling limit. Likewise using equations (4.28), (4.29), (4.30) and (4.35) we have

$$G(x_1, x_2) = W(x_1, x_2)\, dx_1 dx_2 \quad \longrightarrow \quad \tilde{G}(\tilde{x}_1, \tilde{x}_2) = \tilde{W}(\tilde{x}_1, \tilde{x}_2)\, d\tilde{x}_1 d\tilde{x}_2 , \tag{4.40}$$

where $\tilde{G}(\tilde{x}_1, \tilde{x}_2)$ is exactly the 2-point loop correlator on $\Sigma_-$.

So far we have seen that the double points of the near-critical curve do not play a role in taking the limit of the differentials. However, this is not the case for the final two elements of the Feynman rules

$$y\,dx \longrightarrow \sqrt{C}a^{m+n/2+1}\;y_-\,d\tilde{x} \tag{4.41}$$

and

$$\frac{\int_q^x y(s)ds}{y(x)dx} \longrightarrow \frac{\int_{\tilde{q}}^{\tilde{x}} y_-(\tilde{s})d\tilde{s}}{y_-(\tilde{x})d\tilde{x}}\;. \tag{4.42}$$

To summarize: what we have found is that all the relevant quantities reduce to the analogous quantities on the near-critical curve in the limit $a \to 0$. In particular, being careful with the overall scaling, the genus $g$ free energy has the limit

$$F_g \longrightarrow C^{1-g}\Delta^{2-2g}\,F_g(\Sigma_-)\;. \tag{4.43}$$

where we have emphasized that $F_g(\Sigma_-)$ depends only on $\Sigma_-$. This is the result advertised in (2.11) and the property of universality. Similarly, the genus $g$ $p$-point loop functions have the limit

$$W_g(x_1,\ldots,x_p)\,dx_1\cdots dx_p \longrightarrow C^{1-g-p/2}\Delta^{2-2g-p}\,\tilde{W}_g(\tilde{x}_1,\ldots,\tilde{x}_p)\,d\tilde{x}_1\cdots d\tilde{x}_p\;. \tag{4.44}$$

## 5. The physics of the double points

In this section, we consider in more detail the case when the near-critical curve has double points, since as we have already argued the physics is very different from that considered in [4] with only branch points.

It will turn out that the existence of the double points has a profound effect on the physics of the critical region. In the case of branch points, dibaryons become massless when the branch points collide. We want to argue that certain states also become massless when a double point collides with a branch point. In particular the mass scale when $m$ double points and $n$ branch points come together is $Na^{m+n/2+1}$. In order to tease out the physics of these light states let us consider the case with only a single branch point and one double point. The curve in the near critical region has the form

$$y_-^2 = (\tilde{x}-\tilde{z})^2(\tilde{x}-\tilde{b})\;. \tag{5.1}$$

This is precisely the kind of critical curve that appeared in the old matrix literature [46].

In order to motivate the issues involved, let us consider what happens as we go through a transition where the double point arises from the collision of two branch points. This can be achieved by using a more general form of the bare superpotential than the ones we considered in sections 2.2 and 2.3. In this case, on one side of the transition we have three branch cuts ending in the critical region and three light dibaryons associated to the three cycles $C_i$ which link each pair of branch points. Each of the cuts is associated to an abelian gauge field $\mathrm{U}(1)_i$ in the IR gauge group. The situation is illustrated in figure 1. The dibaryon $C_i$ is magnetically, and potentially electrically, charged with respect

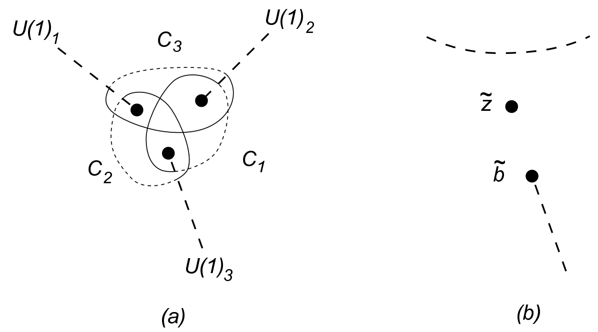


**Figure 1:** (a) A configuration for which $\Sigma_-$ consists of three branch points. (b) Two of the branch points coalesce to form $\Sigma_-$ with a single branch point $\tilde{b}$ and a single double point $\tilde{z}$.

to $\mathrm{U}(1)_j \times \mathrm{U}(1)_k$. In fact, denoting the electric and magnetic charges as $(g_1, q_1; g_2, q_2; g_3, q_3)$ we can choose conventions where the three dibaryons have charges

$$\begin{aligned} C_1 : &\qquad (0,0;1,1;-1,0) \\ C_2 : &\qquad (-1,0;0,-1;1,0) \\ C_3 : &\qquad (1,0;-1,0;0,0)\ . \end{aligned} \tag{5.2}$$

The three dibaryons have mutually non-local charges since as cycles $C_1 \cdot C_2 = C_2 \cdot C_3 = C_3 \cdot C_1 = 1$ and $C_1 + C_2 + C_3 = 0$ [17]. Now suppose two branch points come together corresponding to the vanishing of $C_3$. At this point the dibaryon $C_3$ becomes massless and the double points of the $\mathcal{N} = 1$ and Seiberg-Witten curves coincide. This point in the parameter space touches a new phase where the dibaryon is condensed — the double points of the two curves move apart — and the U(1) subgroup of $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2$ defined by the elements $(e^{i\theta}, e^{-i\theta})$ is confined. This new confined phase is the one we are interested in. Since dibaryons $C_1$ and $C_2$ carry opposite electric charge under the confined U(1) it is natural that $C_1$ and $C_2$ form a bound state: a "meson". The bound state has minus the charges of $C_3$; in other words, the meson is magnetically charged under the confined U(1). Notice, however, that the state is neutral under the two remaining U(1)'s. This means that, in stark contrast to the case of dibaryons, we will not "see" these bound states via the logarithmic running of the coupling of the IR abelian gauge fields as we did for the case of a light dibaryon in section 4. However, as we have seen, their effects will show up in other quantities. In particular, if this meson, of mass $M$, runs around 1-loop graphs then it contributes as $M^{2-2g}$ to the higher-genus free energy $F_g$. So the fact that $F_g \sim \Delta^{2-2g}$ means that since $m = 1$ and $n = 1$ the mass of the meson is $M \sim \Delta = Na^{5/2}$, in this case.

The remaining issue is to argue that the meson becomes massless when the double point coincides with the remaining branch point, since as $a \to 0$ evidently $M \to 0$. One might have thought that the meson has a mass of order the confinement scale and would not become massless at the critical point. The issue is rather reminiscent of the pion in QCD. The pion mass is much lower than the scale of confinement because the pion is the would-be Goldstone boson of chiral symmetry breaking. However, chiral symmetry is explicitly broken by the mass of the light quarks and so a mass for the pion is generated but at a scale much smaller than the confinement scale. In the present situation, when the double point meets the branch point *and* the Seiberg-Witten double point, $z = b = h$, the dibaryon condensate vanishes and the theory is at an $\mathcal{N} = 2$ Argyres-Douglas superconformal field theory [17]. As the Seiberg-Witten double point moves away (but leaving the branch point and double point of the $\mathcal{N} = 1$ curve coincident $h \neq z = b$) the dibaryon condensate develops and $\mathcal{N} = 2$ superconformal symmetry is spontaneously broken. As the Seiberg-Witten double point moves out to infinity, only an $\mathcal{N} = 1$ superconformal symmetry remains. Our proposal is that the mesonic bound-state is the Goldstone mode of this broken symmetry and hence is exactly massless when the branch and double points coincide. As the double point of the $\mathcal{N} = 1$ curve moves away from the branch point, $z \neq b$, this state becomes massive with a mass $\sim \Delta$ as indicated by the behaviour of $F_g$ in the double-scaling limit (4.43).

In the general case, there will be bound states of the type described associated to each pair consisting of a branch point and a double point. These bound states will be magnetically charged with respect to the confined U(1) associated to each double point. In the limit we are considering with $h_i$ finite as $a \to 0$, the confinement scale is always much greater than the masses of the bound states.

## 6. Conclusion

In this paper, we have studied the large-$N$ limit of certain $\mathcal{N} = 1$ theories in the proximity of Argyres-Douglas-type singularities. The exact analysis of the $F$-terms performed via the Dijkgraaf-Vafa matrix model correspondence shows that, close to these singular points, the $1/N$ expansion breaks down. This can be traced back to the appearance of extra light particles of baryonic and/or mesonic nature that become massless at the critical point. As in [4], there is a natural large-$N$ *double-scaling* limit that emerges: namely if one considers approaching the singularity in conjunction with the 't Hooft large-$N$ limit in such a way that the mass $M$ of these baryonic or mesonic states is kept fixed. In this limit, we have argued that the Hilbert space of the theory splits into two mutually decoupled sectors, $\mathcal{H}_+$ and $\mathcal{H}_-$. The sector $\mathcal{H}_+$ obeys the usual large-$N$ scaling and becomes free in the limit, whilst the sector $\mathcal{H}_-$ keeps non-trivial interactions weighted by the effective string coupling $g_{\text{eff}} \sim 1/M$. In [4], it was possible to map this large-$N$ double-scaling limit to an analogous double-scaling limit considered in [12] in the context of the duality between four-dimensional Little String Theories and certain non-critical string backgrounds [15]. This led to the proposal that the non-trivial dynamics of the $\mathcal{H}_-$ sector admits a dual holographic description in terms of the four-dimensional non-critical string. This proposal passes a non-trivial consistency check in that the non-critical string dual exhibits $\mathcal{N} = 2$ supersymmetry while the $F$-term effective action was shown to be consistent with an enhancement from $\mathcal{N} = 1$ to $\mathcal{N} = 2$ supersymmetry in the $\mathcal{H}_-$ sector. In this paper, we have shown that for more general singularities the sector $\mathcal{H}_-$ has only $\mathcal{N} = 1$ supersymmetry. In particular, the effects of the glueball superpotential do not vanish in the limit. Thus, even though the field theory analysis clearly shows that the double-scaling limit still selects a particular sector of the theory which has non-trivial dynamics, we do not identify the would-be non-



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critical string dual as in [4]. For the type of singularities studied in the old matrix model to describe two-dimensional gravity coupled to $c < 1$ conformal field theories, where the near-critical curve has only one branch point along with any arbitrary number of double points, the double-scaling limit describes a sector of a gauge theory with a mass gap and light meson-like composite states which we argued are the approximate Goldstone bosons of superconformal invariance. These cases are special in that there are no massless abelian fields in the $\mathcal{H}_-$ sector; in fact, there is a mass gap.

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## A. Some formulae

In this appendix, we consider the double-scaling limit of various quantities defined on the curve $\Sigma$ (2.4). This is most conveniently done in the basis $\{\tilde{A}_i, \tilde{B}_i\}$ of 1-cycles described in section 3. In particular, for $i \leq [n/2]$ these are cycles on the near-critical curve $\Sigma_-$ in the double-scaling limit.

The key quantities that we will need are the periods

$$M_{ij} = \oint_{\tilde{B}_j} \frac{x^{i-1}}{\sqrt{\sigma(x)}}\, dx\,, \qquad N_{ij} = \oint_{\tilde{A}_j} \frac{x^{i-1}}{\sqrt{\sigma(x)}}\, dx\ . \tag{A.1}$$

First of all, let us focus on $N_{ij}$ where $j \leq [n/2]$, but $i$ arbitrary. By a simple scaling argument, as $a \to 0$,

$$N_{ij} = \int_{b^-_{(j)}}^{b^+_{(j)}} \frac{x^{i-1}}{\sqrt{B(x)}}\, dx \longrightarrow a^{i-n/2} \int_{\tilde{b}^-_{(j)}}^{\tilde{b}^+_{(j)}} \frac{\tilde{x}^{i-1}}{\sqrt{\tilde{B}(\tilde{x})}}\, d\tilde{x} = a^{i-n/2}\, f^{(N)}_{ij}(\tilde{b}_l)\,, \tag{A.2}$$

for some function $f^{(N)}_{ij}$ of the branch points of $\Sigma_-$. Here, $b^{\pm}_{(j)}$ are the two branch points enclosed by the cycle $\tilde{A}_j$. A similar argument shows that $M_{ij}$ scales in the same way:

$$M_{ij} \longrightarrow a^{i-n/2}\, f^{(M)}_{ij}(\tilde{b}_l)\ . \tag{A.3}$$

So both $N_{ij}$ and $M_{ij}$, for $i, j, \leq [n/2]$, diverge in the limit $a \to 0$. On the contrary, by using a similar argument, it is not difficult to see that, for $j > [n/2]$, $N_{ij}$ and $M_{ij}$ are analytic as $a \to 0$ since the integrals are over non-vanishing cycles.

In summary, in the limit $a \to 0$, the matrices $N$ and $M$ will have the following block structure

$$N \longrightarrow \begin{pmatrix} N_{--} & N^{(0)}_{-+} \\ 0 & N^{(0)}_{++} \end{pmatrix}, \qquad M \longrightarrow \begin{pmatrix} M_{--} & M^{(0)}_{-+} \\ 0 & M^{(0)}_{++} \end{pmatrix}, \tag{A.4}$$

where by $-$ or $+$ we denote indices in the ranges $\{1, \ldots, [n/2]\}$ and $\{[n/2]+1, \ldots, s-1\}$ respectively. In (A.4), $N_{--}$ and $M_{--}$ are divergent while the remaining quantities are finite as $a \to 0$.

We also need the inverse $L = N^{-1}$. In the text, we use the polynomials $L_j(x) = \sum_{k=1}^{s-1} L_{jk} x^{k-1}$, which enter the expression of the holomorphic 1-forms associated to our basis of 1-cycles,

$$\oint_{\tilde{A}_i} \omega_j = \delta_{ij} \ . \tag{A.5}$$

These 1-forms are equal to

$$\omega_j(x) = \frac{L_j(x)}{\sqrt{\sigma(x)}}\, dx \ = \frac{\sum_{k=1}^{s-1} L_{jk} x^{k-1}}{\sqrt{\sigma(x)}}\, dx \,, \qquad \oint_{A_i} \omega_j(x) = \delta_{ij} \tag{A.6}$$

where $i, j = 1, \ldots, s-1$. From the behaviour of $N$ in the limit $a \to 0$, we have

$$L = N^{-1} \longrightarrow \begin{pmatrix} N_{--}^{-1} & \mathcal{N} \\ 0 & \left(N_{++}^{(0)}\right)^{-1} \end{pmatrix} , \qquad \mathcal{N} = -N_{--}^{-1}\, N_{-+}^{(0)} \, \left(N_{++}^{(0)}\right)^{-1} \ . \tag{A.7}$$

Since $N_{--}$ is singular we see that $L$ is block diagonal in the limit $a \to 0$. This is just an expression of the fact that the curve factorizes $\Sigma \to \Sigma_- \cup \Sigma_+$ as $a \to 0$. In this limit, using the scaling of elements of $L_{jk}$, we find, for $j \leq [n/2]$,

$$\omega_j \longrightarrow \frac{\sum_{k=1}^{[n/2]} (f^{(N)})^{-1}_{jk}\, \tilde{x}^{k-1}}{\sqrt{\tilde{B}(\tilde{x})}} d\tilde{x} = \tilde{\omega}_j \ . \tag{A.8}$$

the holomorphic 1-forms of $\Sigma_-$. While for $j > [n/2]$,

$$\omega_j \longrightarrow \frac{\sum_{k>[n/2]}^{s-1} (N_{++}^{(0)})^{-1}_{jk}\, x^{k-n/2-1}}{\sqrt{F(x)}} dx \,, \tag{A.9}$$

are the holomorphic 1-forms of $\Sigma_+$.

## References


[1] G. 't Hooft, *A planar diagram theory for strong interactions*, *Nucl. Phys.* **B 72** (1974) 461.

[2] E. Brezin and V.A. Kazakov, *Exactly solvable field theories of closed strings*, *Phys. Lett.* **B 236** (1990) 144;
M.R. Douglas and S.H. Shenker, *Strings in less than one-dimension*, *Nucl. Phys.* **B 335** (1990) 635;
D.J. Gross and A.A. Migdal, *Nonperturbative two-dimensional quantum gravity*, *Phys. Rev. Lett.* **64** (1990) 127.

[3] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200].

[4] G. Bertoldi and N. Dorey, *Non-critical superstrings from four-dimensional gauge theory*, *JHEP* **11** (2005) 001 [hep-th/0507075].

[5] D. Kutasov and N. Seiberg, *Noncritical superstrings*, *Phys. Lett.* **B 251** (1990) 67.

[6] N. Dorey, *S-duality, deconstruction and confinement for a marginal deformation of $N = 4$ SUSY Yang-Mills*, *JHEP* **08** (2004) 043 [hep-th/0310117].

[7] N. Dorey, *A new deconstruction of little string theory*, *JHEP* **07** (2004) 016 [hep-th/0406104].

[8] N. Dorey and T.J. Hollowood, *On the coulomb branch of a marginal deformation of $N = 4$ SUSY Yang-Mills*, *JHEP* **06** (2005) 036 [hep-th/0411163].

[9] T. Friedmann, *On the quantum moduli space of M-theory compactifications*, *Nucl. Phys.* **B 635** (2002) 384 [hep-th/0203256].

[10] F. Cachazo, N. Seiberg and E. Witten, *Phases of $N = 1$ supersymmetric gauge theories and matrices*, *JHEP* **02** (2003) 042 [hep-th/0301006].

[11] A.V. Manohar, *Large-N QCD*, hep-ph/9802419.

[12] A. Giveon and D. Kutasov, *Little string theory in a double scaling limit*, *JHEP* **10** (1999) 034 [hep-th/9909110]; *Comments on double scaled little string theory*, *JHEP* **01** (2000) 023 [hep-th/9911039].

[13] N. Seiberg, *New theories in six dimensions and matrix description of M-theory on $T^5$ and $T^5/\mathbb{Z}_2$*, *Phys. Lett.* **B 408** (1997) 98 [hep-th/9705221];
M. Berkooz, M. Rozali and N. Seiberg, *Matrix description of M-theory on $T^4$ and $T^5$*, *Phys. Lett.* **B 408** (1997) 105 [hep-th/9704089].

[14] O. Aharony, M. Berkooz, D. Kutasov and N. Seiberg, *Linear dilatons, NS5-branes and holography*, *JHEP* **10** (1998) 004 [hep-th/9808149].

[15] A. Giveon, D. Kutasov and O. Pelc, *Holography for non-critical superstrings*, *JHEP* **10** (1999) 035 [hep-th/9907178].

[16] O. Aharony, A. Giveon and D. Kutasov, *LSZ in LST*, *Nucl. Phys.* **B 691** (2004) 3 [hep-th/0404016].

[17] P.C. Argyres and M.R. Douglas, *New phenomena in* SU(3) *supersymmetric gauge theory*, *Nucl. Phys.* **B 448** (1995) 93 [hep-th/9505062].

[18] P.C. Argyres, M. Ronen Plesser, N. Seiberg and E. Witten, *New $N = 2$ superconformal field theories in four dimensions*, *Nucl. Phys.* **B 461** (1996) 71 [hep-th/9511154].

[19] T. Eguchi, K. Hori, K. Ito and S.-K. Yang, *Study of $N = 2$ superconformal field theories in 4 dimensions*, *Nucl. Phys.* **B 471** (1996) 430 [hep-th/9603002].

[20] D. Israel, *Non-critical string duals of $N = 1$ quiver theories*, hep-th/0512166.

[21] I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, *Topological amplitudes in string theory*, *Nucl. Phys.* **B 413** (1994) 162 [hep-th/9307158].

[22] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, *Commun. Math. Phys.* **165** (1994) 311 [hep-th/9309140].

[23] H. Ooguri and C. Vafa, *The c-deformation of gluino and non-planar diagrams*, *Adv. Theor. Math. Phys.* **7** (2003) 53 [hep-th/0302109].


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[24] C. Vafa, *A stringy test of the fate of the conifold*, *Nucl. Phys.* **B 447** (1995) 252 [hep-th/9505023].

[25] R. Dijkgraaf and C. Vafa, *Matrix models, topological strings and supersymmetric gauge theories*, *Nucl. Phys.* **B 644** (2002) 3 [hep-th/0206255].

[26] F. Ferrari, *Quantum parameter space and double scaling limits in $N = 1$ super Yang-Mills theory*, *Phys. Rev.* **D 67** (2003) 085013 [hep-th/0211069].

[27] F. Ferrari, *Exact amplitudes in four dimensional non-critical string theories*, *Nucl. Phys.* **B 617** (2001) 348 [hep-th/0107096].

[28] F. Ferrari, *The large-$N$ limit of $N = 2$ super Yang-Mills, fractional instantons and infrared divergences*, *Nucl. Phys.* **B 612** (2001) 151 [hep-th/0106192].

[29] J. de Boer and Y. Oz, *Monopole condensation and confining phase of $N = 1$ gauge theories via M-theory fivebrane*, *Nucl. Phys.* **B 511** (1998) 155 [hep-th/9708044].

[30] F. Cachazo, K.A. Intriligator and C. Vafa, *A large-$N$ duality via a geometric transition*, *Nucl. Phys.* **B 603** (2001) 3 [hep-th/0103067].

[31] T. Eguchi and Y. Sugawara, *Branches of $N = 1$ vacua and Argyres-Douglas points*, *JHEP* **05** (2003) 063 [hep-th/0305050].

[32] G. Bertoldi, *Matrix models, Argyres-Douglas singularities and double scaling limits*, *JHEP* **06** (2003) 027 [hep-th/0305058].

[33] F. Cachazo, M.R. Douglas, N. Seiberg and E. Witten, *Chiral rings and anomalies in supersymmetric gauge theory*, *JHEP* **12** (2002) 071 [hep-th/0211170].

[34] R. Dijkgraaf and C. Vafa, *On geometry and matrix models*, *Nucl. Phys.* **B 644** (2002) 21 [hep-th/0207106].

[35] R. Dijkgraaf and C. Vafa, *A perturbative window into non-perturbative physics*, hep-th/0208048.

[36] L. Chekhov, A. Marshakov, A. Mironov and D. Vasiliev, *Complex geometry of matrix models*, hep-th/0506075.

[37] I.M. Krichever, *The tau function of the universal whitham hierarchy, matrix models and topological field theories*, hep-th/9205110.

[38] L. Chekhov, A. Marshakov, A. Mironov and D. Vasiliev, *DV and WDVV*, *Phys. Lett.* **B 562** (2003) 323 [hep-th/0301071].

[39] H. Itoyama and A. Morozov, *The Dijkgraaf-Vafa prepotential in the context of general Seiberg-Witten theory*, *Nucl. Phys.* **B 657** (2003) 53 [hep-th/0211245].

[40] F. Cachazo, N. Seiberg and E. Witten, *Chiral rings and phases of supersymmetric gauge theories*, *JHEP* **04** (2003) 018 [hep-th/0303207].

[41] B. Eynard, *Topological expansion for the 1-hermitian matrix model correlation functions*, *JHEP* **11** (2004) 031 [hep-th/0407261].

[42] L. Chekhov and B. Eynard, *Hermitean matrix model free energy: feynman graph technique for all genera*, *JHEP* **03** (2006) 014 [hep-th/0504116].

[43] R. Dijkgraaf, S. Gukov, V.A. Kazakov and C. Vafa, *Perturbative analysis of gauged matrix models*, *Phys. Rev.* **D 68** (2003) 045007 [hep-th/0210238].

[44] G. Bertoldi, *Double scaling limits and twisted non-critical superstrings*, hep-th/0603075.

[45] D. Bessis, *A new method in the combinatorics of the topological expansion*, *Commun. Math. Phys.* **69** (1979) 147;
D. Bessis, C. Itzykson and J.B. Zuber, *Quantum field theory techniques in graphical enumeration*, *Adv. Appl. Math.* **1** (1980) 109.

[46] P. Di Francesco, P.H. Ginsparg and J. Zinn-Justin, *2-d gravity and random matrices*, *Phys. Rept.* **254** (1995) 1 [hep-th/9306153];
P.H. Ginsparg and G.W. Moore, *Lectures on 2-d gravity and 2-d string theory*, hep-th/9304011.

[47] M. Mariño, *Les houches lectures on matrix models and topological strings*, hep-th/0410165.

[48] K. Demeterfi, N. Deo, S. Jain and C.-I. Tan, *Multiband structure and critical behavior of matrix models*, *Phys. Rev.* **D 42** (1990) 4105;
O. Lechtenfeld, R. Ray and A. Ray, *Phase diagram and orthogonal polynomials in multiple well matrix models*, *Int. J. Mod. Phys.* **A 6** (1991) 4491.

[49] E. Witten, *On the structure of the topological phase of two-dimensional gravity*, *Nucl. Phys.* **B 340** (1990) 281;
J. Distler, *2-d quantum gravity, topological field theory and the multicritical matrix models*, *Nucl. Phys.* **B 342** (1990) 523.

[50] L. Molinari, *Phase structure of matrix models through orthogonal polynomials*, *J. Phys.* **A 21** (1988) 1.

[51] R.C. Brower, N. Deo, S. Jain and C.-I. Tan, *Symmetry breaking in the double well hermitian matrix models*, *Nucl. Phys.* **B 405** (1993) 166 [hep-th/9212127].

[52] A. Klemm, M. Marino and S. Theisen, *Gravitational corrections in supersymmetric gauge theory and matrix models*, *JHEP* **03** (2003) 051 [hep-th/0211216].

[53] C. Crnkovic and G.W. Moore, *Multicritical multicut matrix models*, *Phys. Lett.* **B 257** (1991) 322.

[54] D.J. Gross and E. Witten, *Possible third order phase transition in the large-N lattice gauge theory*, *Phys. Rev.* **D 21** (1980) 446;
V. Periwal and D. Shevitz, *Unitary matrix models as exactly solvable string theories*, *Phys. Rev. Lett.* **64** (1990) 1326; *Exactly solvable unitary matrix models: multicritical potentials and correlations*, *Nucl. Phys.* **B 344** (1990) 731.

[55] C. Crnkovic, M.R. Douglas and G.W. Moore, *Loop equations and the topological phase of multi-cut matrix models*, *Int. J. Mod. Phys.* **A 7** (1992) 7693 [hep-th/9108014].

[56] I.R. Klebanov, J.M. Maldacena and N. Seiberg, *Unitary and complex matrix models as 1-d type 0 strings*, *Commun. Math. Phys.* **252** (2004) 275 [hep-th/0309168].

[57] C.R. Nappi, *Painlevé-II and odd polynomials*, *Mod. Phys. Lett.* **A 5** (1990) 2773.

[58] T.J. Hollowood, L. Miramontes, A. Pasquinucci and C. Nappi, *Hermitian versus anti-hermitian one matrix models and their hierarchies*, *Nucl. Phys.* **B 373** (1992) 247 [hep-th/9109046].

[59] M. Sasaki and H. Suzuki, *On matrix realization of random surfaces*, *Phys. Rev.* **D 43** (1991) 4015;
O. Lechtenfeld, *On eigenvalue tunneling in matrix models*, *Int. J. Mod. Phys.* **A 7** (1992) 2335;
J. Jurkiewicz, *Chaotic behavior in one matrix models*, *Phys. Lett.* **B 261** (1991) 260;
D. Senechal, *Chaos in the hermitian one matrix model*, *Int. J. Mod. Phys.* **A 7** (1992) 1491.

[60] Y. Makeenko, *Loop equations in matrix models and in 2-d quantum gravity*, *Mod. Phys. Lett.* **A 6** (1991) 1901.

[61] J. Ambjørn, L. Chekhov, C.F. Kristjansen and Y. Makeenko, *Matrix model calculations beyond the spherical limit*, *Nucl. Phys.* **B 404** (1993) 127 [hep-th/9302014], erratum *ibid.* **B 449** (1995) 681.

[62] G. Akemann, *Higher genus correlators for the hermitian matrix model with multiple cuts*, *Nucl. Phys.* **B 482** (1996) 403 [hep-th/9606004].

[63] J. Ambjørn and G. Akemann, *New universal spectral correlators*, *J. Phys.* **A 29** (1996) L555 [cond-mat/9606129].

[64] R. Dijkgraaf, A. Sinkovics and M. Temurhan, *Matrix models and gravitational corrections*, *Adv. Theor. Math. Phys.* **7** (2004) 1155 [hep-th/0211241].

[65] L. Chekhov, *Genus one correlation to multi-cut matrix model solutions*, *Theor. Math. Phys.* **141** (2004) 1640 [*Teor. Mat. Fiz.* **141** (2004) 358] [hep-th/0401089].